15.4 Green’s theorem

A simple closed curve in plane is one curve \( C, \ r(t) : t \in [a,b] \) such that \( r(a) = r(b) \), and there are no other intersections.

The positive orientation is counterclockwise.

The first version of Green’s theorem:

**Theorem 1.** If \( C \) is a simple closed curve, positively oriented (i.e. counterclockwise oriented) and the region enclosed by it is \( R \), then for any two continuously differentiable functions \( P(x,y) \) and \( Q(x,y) \), we have

\[
\oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dA.
\]

The integral can be written as \( \oint F \cdot dr \), where \( F = \langle P, Q \rangle \). This is the work done on the closed loop.

**Example:** Let \( F = \langle 3xy, 2x^2 \rangle \). \( C \) is the boundary of the region bounded by \( y = x \) and \( y = x^2 - 2x \), oriented counterclockwisely. Evaluate the work done by \( F \) along \( C \) in two ways.

**Solution.** Way 1: we apply Green’s theorem:

\[
W = \oint_C F \cdot dr = \oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dA.
\]

In our case, the two curves intersect at \( (0,0) \) and \( (3,3) \). The region can be written as \( 0 \leq x \leq 3, x^2 - 2x \leq y \leq x \). Further,

\[
Q_x = (2x^2)_x = 4x \\
P_y = (3xy)_y = 3x.
\]

Hence,

\[
W = \int_0^3 \int_{x^2-2x}^x (4x - 3x)dx = \int_0^3 x(3x - x^2)dx = 27 - \frac{81}{4} = \frac{27}{4}.
\]

Way 2 is to integrate the line integral directly. We see \( C = C_1 + C_2 \). \( C_1 \) is the parabola. It can be parametrized as

\[
r(t) = \langle t, t^2 - 2t \rangle, 0 \leq t \leq 3.
\]
Hence,

\[ W_1 = \int_{C_1} (Pdx + Qdy) = \int_0^3 3t(t^2 - 2t)dt + 2t^2(2t - 2)dt \]
\[ = \int_0^3 (7t^3 - 10t^2)dt = \frac{7}{4}t^4 \bigg|_0^3 - \frac{10}{3}t^3 \bigg|_0^3 = \frac{7 \cdot 81}{4} - 90 \]

Let’s now look at the second curve. \( C_2 \) is the line segment. It can be parametrized as

\[ r(t) = (3 - 3t, 3 - 3t), 0 \leq t \leq 1. \]

Then, we have

\[ W_2 = \int_{C_2} (Pdx + Qdy) = \int_0^1 3(3 - 3t)(3 - 3t)(-3dt) + 2(3 - 3t)^2(-3dt) \]
\[ = \int_0^1 (-15) \cdot 9(1 - t)^2dt = 5 \cdot 9(1 - t)^3 \bigg|_0^1 = -45. \]

The total work is

\[ W = W_1 + W_2 = \frac{7 \cdot 81}{4} - 90 - 45 = \frac{567 - 4 \cdot 135}{4} = \frac{27}{4}. \]

We get the same answer using the two ways!  

**Proof and extensions**

*Proof.* Let’s sketch the proof for a convex region \( R \). For other general regions, read the book. If the region is convex, we can write it as \( a \leq x \leq b \) and \( g_1(x) \leq y \leq g_2(x) \). Then,

\[ \int_a^b \int_{g_1(x)}^{g_2(x)} P_y dy dx = \int_a^b (P(x, g_2(x)) - P(x, g_1(x)))dx = -\oint Pdx. \]

For the other term, we can write the region as \( c \leq y \leq d, f_1(y) \leq x \leq f_2(y) \) and the proof is similar.  

Further, if we understand \( \langle P, Q \rangle \) to be the velocity field \( \mathbf{v} \), then \( \int_C \mathbf{v} \cdot d\mathbf{r} \) is also called the circulation.

Green’s theorem says that the circulation equals the integral of curl. The curl is the density of circulation and that is why we relate the curl with rotation!
Corollaries

- The area of region $R$ is given by
  \[ A = -\oint y\,dx = \oint x\,dy = \frac{1}{2} \oint -y\,dx + x\,dy \]

- $\mathbf{F} = \langle P, Q \rangle$. Suppose $D$ has no holes. If $P_y = Q_x$ on $D$, then $\mathbf{F}$ is conservative on $D$, or $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent.

  \[ \text{Proof. } \int_{C_1} - \int_{C_2} = \int_{C_1-C_2} = \oint_C = \iint (Q_x - P_y)\,dA = 0. \]

  \[ \square \]

  \[ \textbf{Example: } \text{Compute the area bounded by x-axis and one arch of the cycloid } r(t) = \langle t - \sin t, 1 - \cos t \rangle. \]

  Use formula above. The answer is $3\pi$. We went over in class. Omitted here.

Generalization

How do we apply Green’s theorems to regions with holes? Regions with several parts?

- If the region is between the outer curve $C_1$ and inner curve $C_2$, then we use the idea of a ‘bridge’ to construct a connected simple curve. The integrals on the bridge is zero.

  \[ \oint_{C_1} - \oint_{C_2} \]

  \[ \textbf{Example: } \mathbf{F} = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle. \text{ Let } C \text{ that is simple and closed, positively oriented. Suppose the origin is not on } C, \text{ compute } \oint_C \mathbf{F} \cdot d\mathbf{r}. \]

  We compute that $P_y = Q_x$ \textbf{except at the origin}. If $(0,0)$ is not inside the curve $C$, we apply Green’s theorem and get the answer to be 0. If the origin is inside, we must use another curve inside to exclude the origin. The curve is whatever you desire. The convenient one is a circle with small radius. In this case, the answer is $2\pi$. The details have been gone over in class.

Second version of Green’s theorem

Flux

We start with the definition of outer flux.
Consider a flow of fluid in the plane with density $\delta$. The velocity field is $v$. $C$ is a curve and $n$ is the unit normal of $C$ such that when it is rotated counterclockwise by $\pi/2$, we have $T$. The net total mass of fluid going across the curve $C$ per unit of time is given by

$$\sum_i \delta_i v_i \cdot n_i \Delta s_i.$$  

Hence, the flux is

$$\Phi = \int_C F \cdot n ds,$$

where $F = \delta v = \langle P, Q \rangle$. Usually, $\delta$ is a constant, and we simply write $\int_C v \cdot n ds$.

Let’s figure out $n$ in 2D: since $T = \frac{1}{|r'(t)|}(x'(t), y'(t))$, then

$$n = T \times k = \frac{1}{|r'(t)|}(y'(t), -x'(t)) = \langle \frac{dy}{ds}, -\frac{dx}{ds} \rangle.$$

Since $ds = |r'(t)|dt$,

$$nds = \langle y'(t), -x'(t) \rangle dt = \langle dy, -dx \rangle.$$

The integral is then written as

$$\Phi = \int_C F \cdot n ds = \int_C Pdy - Qdx.$$

Vector form of Green’s theorem

Let $\tilde{P} = -Q$ and $\tilde{Q} = P$, we then have the first version of Green’s theorem:

$$\int_C F \cdot n ds = \int_C Pdy - Qdx = \int_C \tilde{P} dx + \tilde{Q} dy = \iint_R (\tilde{Q}_x - \tilde{P}_y) dA = \iint_R (P_x + Q_y) dA = \iint_R \nabla \cdot F dA.$$

$$\int_C F \cdot n ds = \iint_R \nabla \cdot F dA.$$

is the vector form of divergence theorem. It says that the flux is equal to the integration of divergence over the region inside.

**Example:** Compute the outer flux of $F = \langle 3xy^2 + 4x, 3x^2y - 4y \rangle$ across the $C$ where $C$ is $y = \sqrt{4 - x^2}, y \geq 0$.

The idea is to construct another path so that the curve is closed. Then, we apply Green’s and take off the part we can compute easily.
Physical meaning of divergence

We apply the Green’s theorem on a circular disk:

\[ \oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_B \nabla \cdot \mathbf{F} dA. \]

If we divide both sides by \( \pi r^2 \) and take \( r \to 0 \), we obtain the following formula:

\[ \nabla \cdot \mathbf{F} = \lim_{r \to 0} \frac{1}{\pi r^2} \oint_C \mathbf{F} \cdot \mathbf{n} ds \]

We know the right hand side is the mass diverging away from the region inside \( C \). In this sense, \( \nabla \cdot \mathbf{F} \) is therefore the net rate at which the fluid is diverging away from point \((x_0, y_0)\), or the density of flux, as we talked in Section 15.1.