15.1 Vector fields

A vector field is a vector-valued function of $x, y$ or $x, y, z$. That means each point in space is associated with a vector. The input is the position vector while the output is some arbitrary vector.

\[
2D : \mathbf{F}(x, y) = P(x, y)i + Q(x, y)j
\]

\[
3D : \mathbf{F}(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k.
\]

Think about: Previously, the parametric surface $\mathbf{r}(u, v)$ is also a vector-valued function. What are the differences among them?

- For parametric surfaces, the input is list of some parameters (which can be regarded as a vector) while the output is the position vector. The collection of terminal points cover the curve or surface.
- For vector field, the input is the position vector while the output is some arbitrary vector, and then we can associate each point in space with a vector.

**Example:** Plot the vector field $\mathbf{F} = -\frac{\mathbf{r}}{r^2}$ where $\mathbf{r} = \langle x, y \rangle$ is the position vector.

The magnitude $1/r$ and the direction is opposite to $\mathbf{r}$.

The gradient vector field

Suppose $f(x, y, z)$ is a differentiable function in space (the input in the position vector while the output is a scalar). Then, the gradient

\[
\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}
\]

is a vector field, because at each point there is a gradient and then the gradient is a vector-valued function of $x, y, z$.

It’s beneficial to introduce the vector differential operator

\[
\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}
\]

which is sometimes called ‘nabla’. The gradient is then the vector field by acting nabla on a scalar function $f$. 

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Physical meaning: The gradient field indicates the fastest increasing direction of the scalar function $f$ at each point. Of course, at different points, the directions are different.

Properties:

- Linearity: $\nabla(af + bg) = a\nabla f + b\nabla g$ for $a, b$ constants.
- Product rule: $\nabla(fg) = g\nabla f + f\nabla g$.

The divergence of a vector field

If we dot nabla with a vector field, we get a scalar output, which is the divergence.

Let $F = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$.

$$\text{div } F = \nabla \cdot F = P_x + Q_y + R_z.$$ 

Physical meaning: The divergence is the density of the field flux. If $\nabla \cdot F > 0$, the flux goes out of this point and if $\nabla \cdot F < 0$, the flux goes into this point. In the former case, we call the point as a source and in the latter case, we call it a sink. (We’ll explain why later using the divergence theorem.)

For example, if the vector field is the velocity field of the fluid, then $\nabla \cdot v < 0$ means the fluid is flowing into one point and this implies that the fluid is being compressed there. Correspondingly, $\nabla \cdot v > 0$ means the fluid is expanding there.

Another example, consider the gravitational field, the divergence of the field is the source of the field. Hence, the mass density. (This claim can be proved rigorously using the inverse square law of gravitational force and divergence theorem later.)

Example: Consider the electronic field

$$E = \langle 3+4x-2y+3x^2-4y^2, 2-2x+2y-z+xy-y^2, 5-x-y-5z+zx-y^2-z^2 \rangle.$$ 

Suppose the charge density is $\rho(x, y, z)$. Compute $\rho(1, 0, 0)/\rho(0, 0, 0)$.

Solution.

$$\rho(1, 0, 0) \rho(0, 0, 0) = \frac{(\nabla \cdot E)(1, 0, 0)}{(\nabla \cdot E)(0, 0, 0)}.$$ 

We figure out $\nabla \cdot E = (4+6x)+(2+x-2y)+(-5+x-2z) = 8x-2y-2z+1$. The ration is therefore $(8 + 1)/1 = 9$. □
**Balance rule:** If a distribution of source generates a flow of a kind of material, at the equilibrium, the density of the source equals the divergence of the flow field. The material source just generates the material. If the strength of the source is bigger than the source of the flux (or the divergence), the material accumulates at that point and there will more and more material there. On the other hand, if the source is less than the source of the flux, the amount of the material at the point will get less and less. Hence, at the equilibrium, they must be equal.

**Useful property:** the product rule:

\[ \nabla \cdot (f \mathbf{F}) = \nabla f \cdot \mathbf{F} + f \nabla \cdot \mathbf{F}. \]

**Example:** Simplify \( \text{div}(\nabla(g)). \)

**Solution.**

\[
\begin{align*}
\nabla \cdot (g \nabla f + f \nabla g) &= \nabla \cdot (g \nabla f) + \nabla \cdot (f \nabla g) = (\nabla g \cdot \nabla f + g \text{div}(\nabla f)) \\
&+ (\nabla f \cdot \nabla g + f \text{div}(\nabla g)) = g \text{div}(\nabla f) + 2 \nabla f \cdot \nabla g + f \text{div}(\nabla g).
\end{align*}
\]

□ Here, \( \text{div}(\nabla f) = \nabla \cdot \nabla f = \nabla \cdot (f_x, f_y, f_z) = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f = \Delta f \) is called the Laplacian of \( f \).

**The curl of a vector field**

If we cross nabla with a vector field, we get another vector field, which is the curl. Let \( \mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle \).

\[
\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
P & Q & R
\end{vmatrix}
\]

****Note: \( \nabla \times \mathbf{F} \) is NOT necessarily perpendicular with \( \mathbf{F} \)****

**Example:** Compute the curl of \( \mathbf{F} = \langle x + \sin(yz), y + \sin(xz), z + \sin(xy) \rangle \).

**Solution.** \( \langle x \cos(xy) - x \cos(xz), y \cos(yz) - y \cos(xy), z \cos(xz) - z \cos(yz) \rangle. \)

□

Physical meaning: The curl is twice of the local angular velocity of the field. In other words, locally, the field can be regarded as a rotation about an axis. The curl then points in the direction of the axis while the magnitude
of the curl is twice of the strength of the rotation speed. If the field is the velocity field of a fluid flow, then it’s called the vorticity.

**Example:** Suppose that the wind field can be described by \( \mathbf{v} = \langle x^2 + 3y^2, \sin(y)z^2, x^3z \rangle \). At point \((1,1,1)\), the air is locally rotating about a line. Find the equation of the straight line.

Properties:

- **product rule:** \( \nabla \times (f \mathbf{F}) = \nabla f \times \mathbf{F} + f \nabla \times \mathbf{F} \).
- **curl of gradient is zero:** \( \nabla \times \nabla f = 0 \).
- **divergence of curl is zero:** \( \nabla \cdot (\nabla \times \mathbf{F}) = 0 \).

**Example:** Prove the third property.

**Proof.** Without loss of generality, assume \( \mathbf{F} = \langle P, Q, R \rangle \).

\[
\nabla \times \mathbf{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y).
\]

Hence,

\[
\nabla \cdot (\nabla \times \mathbf{F}) = (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}
\]

\(R_{yx} = R_{xy}\) by Clairaut’s theorem (recall that \( f_{xy} = f_{yx}\) if they are both continuous). Similarly, the others can be paired. They therefore all cancel out and the result is zero. \(\square\)

**Use vector identities to derive identities for curl and divergence (Omitted)***

There are many interesting identities involving curl and divergence. We can derive them using the double cross product or triple scalar product properties.

**Example:**

By the property \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \), what do you think \( \nabla \times (\mathbf{F} \times \mathbf{G}) \) equals?

Key: one must remember that \( \nabla \) is an operator that must act on both fields by product rule.

For the first, \( (\nabla \cdot \mathbf{G})\mathbf{F} \) only reflects the action on \( \mathbf{G} \). To recover the action on \( \mathbf{F} \), we rewrite \( (\mathbf{G} \cdot \nabla)\mathbf{F} \). Hence, \( (\mathbf{a} \cdot \mathbf{c})\mathbf{b} \) gives \( (\nabla \cdot \mathbf{G})\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} \). (wait! are these two really different? Yes. The first is \( \text{div}(\mathbf{G})\mathbf{F} \) while the second one is \( G_1 \frac{\partial}{\partial x} \mathbf{F} + G_2 \frac{\partial}{\partial y} \mathbf{F} + G_3 \frac{\partial}{\partial z} \mathbf{F} \).
Similarly, the second term also yields two terms. Finally, we have

\[ \nabla \times (F \times G) = (\nabla \cdot G)F + (G \cdot \nabla)F - (\nabla \cdot F)G - (F \cdot \nabla)G \]

The above is not a proof. To prove, you should really compute.

**Example:** What do you think \( \nabla \times (\nabla \times F) \) equals?

In this case, it’s interesting that both operators want to act on \( F \). We can simply apply the identity:

\[
\nabla(\nabla \cdot F) - (\nabla \cdot \nabla)F = \text{grad}(\text{div}(F)) - \Delta F.
\]

**Example:** By the property, \((a \times b) \cdot c = a \cdot (b \times c) = -a \cdot (c \times b)\), what do you think \( \nabla \cdot (F \times G) \) equals?

**Solution.** We change the dot to cross and cross to dot and have \((\nabla \times F) \cdot G\). However, the operator also wants to act on \( G \) and hence we have \(- (F \times \nabla) \cdot G = -F \cdot (\nabla \times G)\). Therefore, we have

\[
\nabla \cdot (F \times G) = (\nabla \times F) \cdot G - F \cdot (\nabla \times G).
\]

**Example:** What do you think \( \nabla \cdot (\nabla f \times \nabla g) \) equals applying the property above?

Applying the identity above, we find it to be 0, since \( \nabla \times (\nabla f) = 0! \)

Nice!

### 15.2 Line integrals

**Review**

Suppose \( C \) is a parametric curve (output is the position vector):

\[ \mathbf{r} = \langle x(t), y(t), z(t) \rangle. \]

We have studied that \( s = \int_0^t |\mathbf{r}'(t)| \, dt \) and

\[ ds = |\mathbf{r}'(t)| \, dt = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt. \]
The line integral of a function

the line integral of a function along a curve $C$ is defined to be

$$\int_C f(x, y, z)ds = \lim_{|\Delta s| \to 0} \sum_{i}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s.$$ 

This means we divide the curve into many segments. Then, we take sample points and construct the Riemann sum. As the partition goes finer and finer, the limit is defined to be the line integral.

Since $\Delta s \approx |r'(t_i^*)|\Delta t$, the integral is

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt.$$ 

This is also called the line integral with respect to arc length.

**Example:** Consider the wire described by $x^2 + y^2 = 4, y \geq 0$. Suppose the density (per unit length) is $\delta = y$. Compute the centroid of the wire and moment of inertia about $x$-axis.

* Solutions have been done in lecture, omitted here.*