14.7 Integration in cylindrical and spherical coordinates

Cylindrical

The Jacobian is

\[
J = \begin{vmatrix}
\cos \theta & -r \sin \theta & 0 \\
\sin \theta & r \cos \theta & 0 \\
0 & 0 & 1
\end{vmatrix} = r.
\]

Hence, \( dV = r \, dr \, d\theta \, dz \).

If we draw a picture, we can see directly that \( dV \) is really \( r \, dr \, d\theta \, dz \).

Spherical

The Jacobian is

\[
J = \begin{vmatrix}
\sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\
\sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\
\cos \phi & -\rho \sin \phi & 0
\end{vmatrix} = \rho^2 \sin \phi.
\]

Hence, \( dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \). If we draw a picture, we can see clearly that this is true.

**Example:** Find the centroid of the first octant portion of the ball \( x^2 + y^2 + z^2 \leq a^2 \) using both cylindrical coordinates and spherical coordinates, assuming the density is uniform.

**Solution.** In Cylindrical way: the sphere is \( r^2 + z^2 = a^2 \). Hence, we can have \( 0 \leq r \leq a, 0 \leq \theta \leq \pi/2, 0 \leq z \leq \sqrt{a^2 - r^2} \). Due to the symmetry, we must have \( \bar{x} = \bar{y} = \bar{z} \). Then,

\[
\bar{z} = \frac{1}{\iiint_T \delta dV} \iiint_T z \delta dV = \frac{1}{\iiint_T dV} \iiint_T zdV
\]

\[
= \frac{1}{\int_0^a \int_0^{\pi/2} \int_0^\sqrt{a^2 - r^2} r \, dz \, d\theta \, dr} \int_0^a \int_0^{\pi/2} \int_0^\sqrt{a^2 - r^2} z \, r \, dz \, d\theta \, dr
\]
In Spherical way: the sphere is $\rho = a$. Hence, $0 \leq \rho \leq a, 0 \leq \phi \leq \pi/2, 0 \leq \theta \leq \pi/2$.

$$V = \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

$$\iiint_T z \, dV = \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} \rho \cos \phi \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} \rho^3 \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta.$$

**Example:** Write out the region bounded by $z = x^2 + y^2$ and $z = y$ in cylindrical coordinates.

**Solution.** We have done this example before. In cylindrical, they are $z = r^2$ and $z = r \sin \theta$. The intersection is $r = \sin \theta$. The projection onto $xy$ plane is a circle. For $\theta$, we set $r = 0$, and see $0, \pi$ are two adjacent zeros. Hence, $0 \leq \theta \leq \pi, 0 \leq r \leq \sin \theta, r^2 \leq z \leq r \sin \theta$.

**Example:** Set up the integral for the area inside the two circles $r = 1$ and $r = 2 \sin \theta$. Set up the integral for the volume of the solid bounded by $r = 1, r = 2 \sin \theta, z = y$ and the $xy$ plane.

**Solution.** For the area, $A = \iint_R dA$. In polar, $dA = r \, dr \, d\theta$. We see that we must divide the integral into three pieces. $1 = 2 \sin \theta$. We find $\theta = \pi/6$ and $\theta = 5\pi/6$. Hence,

$$A = \int_0^{\pi/6} \int_0^{2 \sin \theta} r \, dr \, d\theta + \int_{\pi/6}^{5\pi/6} \int_0^1 r \, dr \, d\theta + \int_{5\pi/6}^{\pi} \int_0^{2 \sin \theta} r \, dr \, d\theta.$$

The volume is $V = \iiint_R (z_2 - z_1) \, dA = \iiint_R y \, dA$. Hence,

$$V = \int_0^{\pi/6} \int_0^{2 \sin \theta} r \sin \theta r \, dr \, d\theta + \int_{\pi/6}^{5\pi/6} \int_0^1 r \sin \theta r \, dr \, d\theta + \int_{5\pi/6}^{\pi} \int_0^{2 \sin \theta} r \sin \theta r \, dr \, d\theta.$$

**Example:** Set up the integral for the volume bounded by $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 - 2x = 0$. 

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Solution. We use cylindrical coordinates. \( r^2 + z^2 = 4 \) and \( r - 2 \cos \theta = 0 \). The region is determined by \( r - 2 \cos \theta = 0 \). We set \( r = 0 \) and have \( \cos \theta = 0 \). Hence, \( -\pi/2 \leq \theta \leq \pi/2 \). Then, \( 0 \leq r \leq 2 \cos \theta \). For \( z \), we find \( -\sqrt{4 - r^2} \leq z \leq \sqrt{4 - r^2} \). Note \( dV = rdrd\theta dz \). The volume is therefore:

\[
V = \iiint dV = \int_{-\pi/2}^{\pi/2} \int_{0}^{2 \cos \theta} \int_{\sqrt{4-r^2}}^{\sqrt{4-r^2}} r dzdrd\theta
\]

Example: Set up the integral for the mass of the region contained in the sphere \( x^2 + y^2 + (z - a)^2 = a^2 \) but below \( z = r \) with unit density.

If we draw the picture, we see that the spherical coordinates are the best.

Solution. \( z = r \) is just \( \phi = \pi/4 \). The sphere is \( \rho^2 - 2a \rho \cos \phi = 0 \) or \( \rho = 2a \cos \phi \). Hence, we have \( \pi/4 \leq \phi \leq \pi/2, 0 \leq \rho \leq 2a \cos \phi, 0 \leq \theta < 2\pi \). Then,

\[
m = \int_{\pi/4}^{\pi/2} \int_{0}^{2a \cos \phi} \int_{0}^{2\pi} 1 \ast \rho^2 \sin \phi d\theta d\rho d\phi.
\]

Example: Consider the ice-cream cone above \( \phi = \pi/6 \) but below \( \rho = 2a \cos \phi \). Suppose the density is \( \delta = 1 \). Set up the integrals for the total mass and centroid.

Solution. This problem is convenient in spherical coordinates. \( 0 \leq \phi \leq \pi/6, 0 \leq \theta < 2\pi, 0 \leq \rho \leq 2a \cos \phi \). The volume element is \( dV = \rho^2 \sin \phi d\rho d\theta d\phi \).

The total mass is

\[
m = \iiint dV = \int_{0}^{\pi/6} \int_{0}^{2\pi} \int_{0}^{2a \cos \phi} 1 \rho^2 \sin \phi d\rho d\theta d\phi.
\]

For the centroid, we use symmetry do conclude that \( \bar{x} = \bar{y} = 0 \). Then,

\[
\bar{z} = \frac{1}{m} \iiint dV = \frac{1}{m} \int_{0}^{\pi/6} \int_{0}^{2\pi} \int_{0}^{2a \cos \phi} \rho \cos \phi \rho^2 \sin \phi d\rho d\theta d\phi
\]

because \( z = \rho \cos \phi \).
14.8 Surface Area

Previously, we see that a vector valued function with a single variable \( r(t) \) is a curve in space.

Now, if the function is vector-valued but has two variables (parameters)

\[
r(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle,
\]

where \( r \) again is the position vector of the point , what will the object be?

Fixing \( v = v_0 \), \( r(u, v_0) \) is a curve. Now, for different \( v = v_1 \), it’s another curve. The object is thus a family of curves, and they form a surface.

**Example:** The graph \( z = f(x, y) \) is a surface in 3D space. Parametrize it.

\[
r = \langle x, y, z \rangle. \text{ We choose } x, y \text{ as the parameters. Then, } r = \langle x, y, f(x, y) \rangle.
\]

**Example:** Let \( \rho, \phi, \theta \) be the spherical coordinates. The function \( \rho = h(\phi, \theta) \) gives a surface in the space. (Example is \( \rho = 2 \).) Parametrize this surface.

**Solution.** \( r = \langle x, y, z \rangle = \langle h(\phi, \theta) \sin \phi \cos \theta, h(\phi, \theta) \sin \phi \sin \theta, h(\phi, \theta) \cos \phi \rangle \).

**Example:** Parametrize the rectangle \( 0 \leq x \leq 2, 0 \leq y \leq 3, z = 1 \).

**Solution.** This is a special case of the first example, \( z = f(x, y) = 1 \) and hence

\[
r(x, y) = \langle x, y, 1 \rangle, \quad 0 \leq x \leq 2, 0 \leq y \leq 3
\]

Given a parametric surface

\[
r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k = \langle x(u, v), y(u, v), z(u, v) \rangle,
\]

we call it smooth if

\[
r_u = \langle x_u, y_u, z_u \rangle, \quad r_v = \langle x_v, y_v, z_v \rangle,
\]

are both nonzero and nonparallel.

Consider the small area for the rectangle \( \Delta u \Delta v \) in \( u-v \) plane. It’s a parallelogram on the surface under the mapping \( r(u, v) \). **Draw a picture.**

One edge is \( a = r(u + \Delta u, v) - r(u, v) \approx r_u \Delta u \). The other edge is similarly \( b \approx r_v \Delta v \). The area is therefore

\[
\Delta S \approx |a \times b| \approx |r_u \times r_v| \Delta u \Delta v.
\]
\( \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \) is a normal vector of the surface. The total area is

\[
A = a(S) = \iint_{u,v} dS = \iint_{u,v} |\mathbf{r}_u \times \mathbf{r}_v| dudv.
\]

\( dS = |\mathbf{r}_u \times \mathbf{r}_v| dudv \)

is the surface area element and \( d\vec{S} = \mathbf{N} |\mathbf{N}| dS = \mathbf{N} dudv = \mathbf{r}_u \times \mathbf{r}_v dudv \) is the directed surface area element.

Here, we see that \( |\mathbf{r}_u \times \mathbf{r}_v| \) plays the same role as the Jacobian in the change of variables for double integrals. It’s the amplification factor between the areas.

**Example:** If \( u, v \) are the Cartesian coordinates \( x, y \), then \( \mathbf{r} = \langle x, y, f(x, y) \rangle \). It is the graph of \( z = f(x, y) \). \( \mathbf{r}_x \times \mathbf{r}_y = \langle -f_x, -f_y, 1 \rangle \). This makes sense as it is just \( \nabla F \) where \( F = z - f(x, y) \). The area is

\[
A = \iint_R \sqrt{1 + f_x^2 + f_y^2} dxdy.
\]

We compute the area of the ellipse cut from \( z = 2x + 2y + 1 \) by \( x^2 + y^2 = 1 \).

What if the surface is the one cut from \( x = 2y + 2z + 1 \) from \( y + z = 1, y = 0, z = 0 \)?

**Example:** Find the area of the spiral ramp \( z = \theta, 0 \leq r \leq 1, 0 \leq \theta \leq \pi \).

**Solution.** We parametrize the surface \( \mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, \sqrt{3} r \rangle \).

\( \mathbf{r}_r \times \mathbf{r}_\theta = \langle \cos \theta, \sin \theta, 0 \rangle \times \langle -r \sin \theta, r \cos \theta, 1 \rangle \)

The magnitude of this is \( \sqrt{1 + r^2} \)

The integral is

\[
\int_0^1 \int_0^\pi \sqrt{1 + r^2} d\theta dr = \pi \int_0^{\pi/4} \sec^3 \theta d\theta = \frac{\pi}{2} (\sqrt{2} + \ln(1 + \sqrt{2})) \]

\( \square \)

**Exercise.** Compute the area of the portion of \( z^2 = 3(x^2 + y^2) \) below \( z = 3 \), and above \( xy \) plane.

\( \mathbf{r} = \langle r \cos \theta, r \sin \theta, \sqrt{3} r \rangle \). \( 0 \leq r \leq \sqrt{3}, 0 \leq \theta < 2\pi \). The magnitude of \( \mathbf{r}_r \times \mathbf{r}_\theta \) is \( 2r \). Then \( \int_0^{\sqrt{3}} \int_0^{2\pi} 2r d\theta dr \)

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