14.9 (Part 2) Change of variables in triple integrals

Suppose we have the transform \( x = x(u, v, w), \ y = y(u, v, w) \) and \( z = z(u, v, w) \). Similarly, by the transformation, a small rectangular box is changed to a small parallelepiped. By computing the volume of the small parallelepiped using triple product, we find that the amplification factor from the volume \( du \ dv \ dw \) to the volume in \( xyz \) space is given by \( |J| \) where the Jacobian is given by

\[
J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix}
  x_u & x_v & x_w \\
  y_u & y_v & y_w \\
  z_u & z_v & z_w
\end{vmatrix}
\]

The triple integral is then equation to

\[
\int \int \int_T f(x, y, z) \, dV = \int \int \int_D f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw
\]

\[
= \int \int \int_D f(x, y, z) \left( \frac{1}{\left| \frac{\partial(u, v, w)}{\partial(x, y, z)} \right|} \right) \, du \, dv \, dw
\]

Depending on which Jacobian is convenient, you can choose the suitable way to evaluate.

**Example:** Find the volume of the ellipsoid \( \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \leq 1 \).

**Solution.** We see if we do \( u = x/2, v = y/3, w = z/4 \), then the region becomes a ball, which would be easy. The Jacobian is

\[
J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix}
  2 & 0 & 0 \\
  0 & 3 & 0 \\
  0 & 0 & 4
\end{vmatrix} = 2 \times 3 \times 4.
\]

Hence, the volume is

\[
V = \int \int \int_T dV = \int \int \int_{u^2 + v^2 + w^2 \leq 1} |J| \, du \, dv \, dw
\]

\[
= \int \int \int_{u^2 + v^2 + w^2 \leq 1} 2 \times 3 \times 4 \, du \, dv \, dw = 2 \times 3 \times 4 \times V(\text{unit ball}) = 2 \times 3 \times 4 \times \frac{4\pi}{3} \times 1^3.
\]
**Example:** Go over it if we have time. Find the volume of the solid torus obtained by revolving the disk \((x-b)^2 + z^2 \leq a^2\) in the \(xz\) plane about the \(z\)-axis.

Previously, we saw that the volume is \(V = 2\pi^2 a^2 b\) by Pappus’s theorem. Now let’s compute this using triple integral.

**Solution.** We first parametrize the torus. Suppose for a point \((x,y,z)\), the corresponding center is at \((b \cos u, b \sin u, 0)\) and the distance between them is \(w\). Assume the angle of the line segment from the \(xy\) plane is \(v\). Then, \(z = w \sin v\). The projection of the point onto the \(xy\) plane is \(b + w \cos v\) away from the origin. Hence, \(x = (b + w \cos v) \cos u, y = (b + w \cos v) \sin u\).

\[ x = (b + w \cos v) \cos u, \quad y = (b + w \cos v) \sin u, \quad z = w \sin v, \quad 0 \leq w \leq a, \quad 0 \leq u, v < 2\pi. \]

The Jacobian is given by

\[
J = \frac{\partial (x,y,z)}{\partial (u,v,w)} = \begin{vmatrix} (b + w \cos v)(-\sin u) & -w \sin v \cos u & \cos v \cos u \\ (b + w \cos v) \cos u & -w \sin v \sin u & \cos v \sin u \\ 0 & w \cos v & \sin v \end{vmatrix} = (b + w \cos v)w.
\]

hence, the volume is

\[
V = \int_0^a \int_0^{2\pi} \int_0^{2\pi} |J| du dv dw = \int_0^a \int_0^{2\pi} \int_0^{2\pi} (b+w \cos v)wdudvdw = 2\pi^2 a^2 b.
\]

\[
\square
\]

12.8. Cylindrical and spherical coordinates

For triple integrals, we usually use two kinds of special coordinates.

**Cylindrical coordinates**

Polar+\(z\) coordinates to represent any point in space: \((r, \theta, z)\). **Draw a picture.**

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.
\]

\[
r = \sqrt{x^2 + y^2}, \quad \tan(\theta) = y/x, \quad z = z.
\]
Spherical coordinates

**Draw a picture.**

We use the distance to the origin $\rho$, the angle measured from $z$-axis $\phi$, and the polar angle $\theta$ (sometimes it’s called azimuthal angle in the spherical coordinate case). We have $(\rho, \phi, \theta)$.

Hence, we have

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.$$  

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \cos \phi = z/\rho, \quad \tan \theta = y/x.$$  

Further, we also have:

$$r = \sqrt{x^2 + y^2} = \rho \sin \phi.$$  

**Comment:** The names for the angles are different from the common ones in physics and engineering. In physics and engineering, many people use $\theta$ to mean the angle measured from $z$-axis and use $\varphi$ to mean the azimuthal angle: $(\rho, \theta, \varphi)$.

**Examples**

**Example:** Write out the region $x^2 + y^2 + z^2 \leq a^2$ in both cylindrical and spherical coordinates.

**Solution.** In cylindrical: the boundary surface is $r^2 + z^2 = a^2$. $0 \leq r \leq a, 0 \leq \theta < 2\pi, -\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2}$.

In spherical, the boundary surface is $\rho = a$. $0 \leq \rho \leq a, 0 \leq \phi \leq \pi, 0 \leq \theta < 2\pi$. □

**Example:** Sketch the solid bounded by $z = r^2$ and $z = 8 - r^2$ where $r$ is the first cylindrical coordinate.

**Solution.** Both surfaces are surfaces of revolution about $z$ axis. They are radially symmetric.

When they intersect, $r^2 = 8 - r^2$ or $r = 2$.

$0 \leq r \leq 2, 0 \leq \theta < 2\pi, r^2 \leq z \leq 8 - r^2$. □

**Example:** Describe the surface $\rho = 2 \cos \phi$.

**Solution.** This is $\sqrt{x^2 + y^2 + z^2} = 2z/\sqrt{x^2 + y^2 + z^2}$ or $x^2 + y^2 + z^2 = 2z$. This is the sphere centered at $(0, 0, 1)$ with radius 1. □
14.7 Integration in cylindrical and spherical coordinates

Cylindrical

The Jacobian is
\[ J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix}
\cos \theta & -r \sin \theta & 0 \\
\sin \theta & r \cos \theta & 0 \\
0 & 0 & 1
\end{vmatrix} = r. \]

Hence, \( dV = r dr d\theta dz \).

If we draw a picture, we can see directly that \( dV \) is really \( r dr d\theta dz \).

Spherical

The Jacobian is
\[ J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix}
\sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\
\sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\
\cos \phi & -\rho \sin \phi & 0
\end{vmatrix} = \rho^2 \sin \phi. \]

Hence, \( dV = \rho^2 \sin \phi d\rho d\phi d\theta \). If we draw a picture, we can see clearly that this is true.

**Example:** Set up the integral for the moment of inertia \( I_x \) for the unit ball \( x^2 + y^2 + z^2 \leq 1 \) with density \( \delta = 1 \) in spherical coordinates.