Math 212-Lecture 13

14.9 (Part 1) Change of variables in double integrals

Recall that for one variable integral
\[ \int f(x)dx = \int f(g(u))g'(u)du. \]
Here, \( g'(u) \) is the ‘amplification factor’ that relates \( dx \) with \( du \). If \( u \) has a change \( du \), then the change of \( x = g(u) \) is \( dx = g'(u)du \).

For double integral, suppose \( x = x(u,v) \) and \( y = y(u,v) \), or \( t = (x(u,v), y(u,v)) \). Then, \( t \) is a transformation from \( uv \) plane to \( xy \) plane. **(Draw the picture and show)**.

How do we find the ‘amplification factor’?

Suppose we have a small rectangle with area \( \Delta u \Delta v \) in the \( uv \) plane. Correspondingly, what is the area in the \( xy \) plane? It is a parallelogram. One edge is given by the vector \( \langle x(u + \Delta u, v), y(u + \Delta u, v) \rangle - \langle x(u, v), y(u, v) \rangle \approx (x_u i + y_u j) \Delta u = t_u \Delta u \).

Similarly, the other edge is given by
\[ t_v \Delta v \approx (x_v i + y_v j) \Delta v. \]

The area in the \( xy \) plane is therefore \( \Delta A = |t_u \times t_v| \Delta u \Delta v \). Hence, the amplification factor is
\[ |t_u \times t_v| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \]

We call
\[ J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}, \]
the Jacobian of the transformation. In other words, the amplification factor is the absolute value of the Jacobian.

Hence, the double integral can be evaluated as
\[ \iint_R f(x,y)dA = \iint_R f(x,y)dxdy = \iint_D f(x(u,v), y(u,v)) | \frac{\partial(x,y)}{\partial(u,v)} | dudv. \]

Facts:
\[ \frac{\partial(x,y)}{\partial(u,v)} \bigg|_{x=x(u,v), y=y(u,v)} = 1 \]
The double integral can also be computed using
\[ \int\int_{R} f(x, y) \, dA = \int\int_{D} f(x, y) \frac{1}{|\frac{\partial(u, v)}{\partial(x, y)}|} \, dvdu. \]

**Example:** Find the area of the region bounded by \( y = \frac{1}{x}, x = \frac{2}{y}, y = 2x^2 \) and \( y = 4x^2 \).

**Solution.** We do change of variables: \( u = xy \) and \( v = y/x \). Then, the region becomes \( 1 \leq u \leq 2, 2 \leq v \leq 4 \), which is a rectangle in \( uv \) plane.

From here, \( y = u/x \) and hence \( v = u/(x^3) \) or \( x = (u/v)^{1/3} = u^{1/3}v^{-1/3} \). \( y = u^{2/3}v^{1/3} \). Then, the Jacobi is

\[ J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{3}u^{-2/3}v^{-1/3} & -\frac{1}{3}u^{1/3}v^{-4/3} \\ \frac{1}{3}u^{-1/3}v^{-1/3} & \frac{1}{3}u^{2/3}v^{-2/3} \end{vmatrix} = \frac{1}{9}v^{-1} + \frac{2}{9}v^{-1} = \frac{1}{3}v^{-1}. \]

Hence, the area is

\[ A = \int_{1}^{2} \int_{2}^{4} |J| \, dvdu = \int_{1}^{2} \int_{2}^{4} \frac{1}{3v} \, dvdu = \int_{1}^{2} \frac{1}{3} \ln(4/2) \, du = \frac{1}{3} \ln 2. \]

Another way to compute the Jacobian is to find

\[ \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ -2\frac{u}{x^2} & \frac{1}{x^2} \end{vmatrix} = 3\frac{y}{x^2} = 3v. \]

Then, the Jacobian we want is the inverse of it, or \( \frac{1}{3v} \). \( \Box \)

**Example:** Evaluate the integral \( \int\int_{R} \frac{1}{(x^2 + y^2)^2} \, dA \) where \( R \) is the one bounded by \( x^2 + y^2 = 6x, x^2 + y^2 = 8y, x^2 + y^2 = 2y \) using the change of variables \( u = 2x/(x^2 + y^2) \) and \( v = 2y/(x^2 + y^2) \).

We’ll do this using the inverse instead of solving out \( x, y \) explicitly.

### 10.2 Polar coordinates

Let \( r \) be the distance of \( (x, y) \) to the origin and \( \theta \) be the angle measured from positive \( x \)-axis. Then, \( (r, \theta) \) can describe a point in the plane uniquely as well.

To locate the point, we first of all find the ray with angle \( \theta \) and then use the distance \( r \) to locate the point.

In general \( r \) can be negative, and \( \theta \) can be added with a multiple of \( 2\pi \). For example, \((2, 7\pi/6)\) and \((-2, \pi/6)\) will be the same point.
For convenience, we usually use \( r \geq 0 \). To cover the full plane once, we can do \( r \geq 0 \) and \( 0 \leq \theta < 2\pi \).

Clearly, we have the following relations if we use the convention \( r \geq 0 \):

\[
x = r \cos \theta, \quad y = r \sin \theta.
\]

and

\[
r = \sqrt{x^2 + y^2}, \quad \tan \theta = y/x. (x \neq 0)
\]

**Example:** Find a pair of polar coordinates for \((x, y) = (-1, \sqrt{3})\).

\((r, \theta) = (2, 2\pi/3)\)

**Example:** Find the polar equation for the circle centered at \((1/2, 0)\) with radius \(1/2\).

**Solution.** The circle in Cartesian coordinates is \((x - 1/2)^2 + y^2 = (1/2)^2\) or \(x^2 - x + y^2 = 0\). Plug in \(x = r \cos \theta, y = r \sin \theta\), we have \(r^2 - r \cos \theta = 0\), or \(r = \cos \theta\). □

**Example:** Express the right half of the disk centered at \((x, y) = (0, 0)\) with radius 2 using polar coordinates.

By the picture, we see directly that \(0 \leq r \leq 2, -\pi/2 \leq \theta \leq \pi/2\)

**Example:** Express \(r = \sin(2\theta)\) in Cartesian coordinates.

We have \(r = 2 \sin \theta \cos \theta\). Using \(\sin \theta = y/r = y/\sqrt{x^2 + y^2}\) and \(\cos \theta = x/r = x/\sqrt{x^2 + y^2}\), we have

\[
\sqrt{x^2 + y^2} = 2 \frac{xy}{x^2 + y^2},
\]

or \((x^2 + y^2)^3 = 4x^2y^2\). If we don’t include negative \(r\), we should impose \(xy \geq 0\).

### 14.4 Double integrals in polar coordinates

We consider the amplification factor for the transformation \(x = r \cos \theta, y = r \sin \theta\). The Jacobian is

\[
J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r.
\]

This tells us that \(dA = rdrd\theta\) (where \(dA\) is the area element in \(xy\) plane.)

**By the picture** (show this in class), we can determine that the area is \(dA = rdrd\theta\) directly. Hence, we can do \(dxdy \rightarrow rdrd\theta\).
**Example:** Evaluate the integral:
\[
\int_0^2 \int_{\sqrt{4-x^2}}^{-\sqrt{4-x^2}} e^{-x^2-y^2} dy \, dx.
\]

**Solution.** The region is \(0 \leq x \leq 2\) and \(-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}\). This is the right half of the disk centered at \((0,0)\) with radius 2.

We use the polar coordinates. \(0 \leq r \leq 2\) and \(-\pi/2 \leq \theta \leq \pi/2\). \(dy \, dx \rightarrow r \, dr \, d\theta\). We have
\[
\int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} r \, dr \, d\theta = \pi \int_0^2 r e^{-r^2} \, dr = -\frac{\pi}{2} e^{-r^2} \bigg|_0^2 = \frac{\pi}{2} (1 - e^{-4}).
\]

Polar coordinates is convenient for **radially simple** region \(\alpha \leq \theta \leq \beta, r_1(\theta) \leq r \leq r_2(\theta)\).

**Example:** Let’s evaluate the volume of the solid bounded by \(z = x^2 + y^2\) and \(z = y\).

**Solution.** Previously (see the lecture notes for volumes using double integrals), we agreed that the volume is
\[
V = \iint_R (y - x^2 - y^2) \, dA
\]
where \(R\) is the region bounded by the circle \(x^2 + y^2 = y\). This equation is \(r = \sin \theta\) in polar coordinates.

Letting \(r = 0\), we have \(\theta = 0, \pi\) (note that \(0 \to 2\pi\) will cover the disk twice.) hence, the region is \(0 \leq \theta \leq \pi, 0 \leq r \leq \sin \theta\). The integral becomes
\[
\int_0^\pi \int_0^{\sin \theta} (r \sin \theta - r^2) r \, dr \, d\theta = \int_0^\pi \frac{1}{12} \sin^4 \theta \, d\theta.
\]
We are integrating even powers of \(\sin \theta\). We do \(\sin^2 \theta = (1 - \cos(2\theta))/2\). Then,
\[
\frac{1}{12} \int_0^\pi \frac{1}{4} [1 + \cos^2(2\theta)] d\theta,
\]
since the integral of \(-2 \cos 2\theta\) is zero. Lastly, \(\cos^2(2\theta) = (1 + \cos(4\theta))/2\). The final answer is \(\pi/32\) \(\square\)

**Remark 1.** We have a fact: if \(f(x,y) = g(x)g(y)\), then \(\int_b^a \int_a^b f(x,y) \, dA = (\int_a^b g(x) \, dx)^2\).

**Example:** Evaluate \(I = \int_0^\infty e^{-x^2} \, dx\). 

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Solution. \( I = \int_0^\infty e^{-x^2} \, dx = \int_0^\infty e^{-y^2} \, dy \). hence,

\[
I^2 = \int_0^\infty e^{-x^2} \, dx \int_0^\infty e^{-y^2} \, dy = \iint_R e^{-x^2-y^2} \, dA,
\]

where \( R \) is the first quadrant. In polar, \( 0 \leq r < \infty \) and \( 0 \leq \theta \leq \pi/2 \). Then,

\[
I^2 = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r \, dr \, d\theta = \frac{\pi}{4}.
\]

hence, \( I = \sqrt{\pi}/2 \). \( \square \)

Exercise. Set up the integral for the volume under \( f(x, y) = x^2 \) and above the region \( \mathcal{D} = \{(x, y) : x^2 + y^2 \leq 4, x^2 + (y - 2)^2 \leq 4\} \) using polar coordinates.