14.3 Areas, volumes and double integrals

Areas of a region in 2D plane

Recall that the area between two function graphs in the \(xy\) plane is given by:

\[
A = \int_{a}^{b} (f_2(x) - f_1(x)) \, dx.
\]

Using the double integral, it’s clear that

\[
A = \iint_{R} \, dA.
\]

The second formula is true by the Riemann sums: we sum up all the areas of the small regions and clearly it will be the total area. The first formula actually can be derived from the second if the region is vertically simple \(a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\).

**Example:** Find the area of the region bounded by \(x = 2y^2 - 1\) and \(x = y^4\).

Volumes in 3D space

Recall that \(\iint_{R} f(x, y) \, dA\) is the volume under \(z = f(x, y)\) and above \(xy\) plane over the region \(R\). If \(f = 1\), the volume equals the base area times 1. Hence, \(\iint_{R} 1 \, dA\) is the area of \(R\). This is another explanation of the formula above.

How do we compute a volume between two surfaces \(z_1 = f_1(x, y)\) and \(z_2 = f_2(x, y)\)? Imagine the Riemann sum again, we find

\[
V = \iint_{R} (z_2 - z_1) \, dA.
\]

As you can imagine, the volume can also be evaluated by the triple integral

\[
V = \iiint_{T} \, dV;
\]

we’ll come back later.

**Example:** Find the volume of the solid bounded by \(z = 6\), \(z = 2y\), \(y = x^2\) and \(y = 2 - x^2\).
Solution. Here, \( y = x^2 \) and \( y = 2 - x^2 \) are two cylinders with rulings parallel with \( z \). Hence, we only need to look at their traces in \( xy \) plane. The intersection of them \( x^2 = 1 \) or \( x = \pm 1 \), \( y = 1 \). For the two planes, clearly \( z = 2y \) is below \( z = 6 \) over this region. Hence, the volume is

\[
\int_{-1}^{1} \int_{x^2}^{2-x^2} (6 - 2y)dydx = \frac{32}{3}.
\]

Example: Set up the integral for the volume of the solid bounded by \( z = x^2 + y^2 \) and \( z = y \). (We’ll evaluate it later using polar coordinates.)

Solution. \( z = x^2 + y^2 \) is the paraboloid revolved from \( z = x^2, y = 0 \) about \( z \)-axis. We see that \( z = x^2 + y^2 \) is below \( z = y \) for the region. hence, the volume is

\[
V = \iint_R (y - (x^2 + y^2))dA.
\]

Let’s determine the region. The intersection is \( x^2 + y^2 = y, z = y \). The projection onto \( xy \) plane is \( x^2 + y^2 = y \). This is a circle centered at \((0, 1/2)\) with radius \( 1/2 \). Hence, the region is the one bounded by this circle. We’ll evaluate this integral later.

14.5 Applications of Double integrals: centroid(center of mass), moments of inertia

Centroid

Suppose a lamina (thin plate) has a surface density \( \delta \) (unit mass per unit area). Then, the total mass is

\[
m \approx \sum_i \delta(x_i, y_i) \Delta A_i \rightarrow m \approx \iint_R \delta(x, y)dA.
\]

The centroid or center of mass \((\bar{x}, \bar{y})\) is defined to be

\[
\bar{x} = \frac{1}{m} \iint_R x\delta(x, y)dA, \quad \bar{y} = \frac{1}{m} \iint_R y\delta(x, y)dA.
\]

These are the average values of \( x \) and \( y \) with respect to mass. In other words, they are the weighted averages of \( x \) and \( y \) with weights to be the mass.
Example: Set up the integrals for the centroid of the upper half disk $x^2 + y^2 \leq a^2, y \geq 0$ with density $\delta = \sqrt{x^2 + y^2}$.

Here, for $\bar{x}$, we can take advantage of the symmetry principle: both the region and the density are symmetric about $y$-axis. Hence, the centroid must lie on the $y$-axis and $\bar{x} = 0$.

$$\bar{y} = \frac{\iint_R y \delta(x, y) dA}{\iint_R \delta(x, y) dA} = \frac{\iint_R y \sqrt{x^2 + y^2} dA}{\iint_R \sqrt{x^2 + y^2} dA}.$$  

After we have studied integration in polar coordinates, we can evaluate this easily.

A typical application of centroid is the theorems of Pappus:

**Theorem 1.** The volume of the solid of revolution equals the area of the cross section times the circumference/perimeter of the circle for the centroid (with uniform density),

$$V = A * d.$$  

For the proof, we divide the region into many small pieces. Each generates a slim ‘doughnut’. The volume of this slim doughnut is given by $\Delta V \approx 2\pi x \Delta A$. Hence, the total volume is

$$V = \iint 2\pi x dA = A(\frac{1}{A} \iint x dA) = A * d.$$  

**Example:** If we revolve the disk $(x-1)^2 + y^2 = 1/4$ about $z$-axis. We get a torus. What is the volume of the torus?

Similarly, we can consider the area of surface of revolution. We define the centroid of the curve $C$ to be

$$\bar{x} = \frac{1}{s} \int_C x \delta ds, \quad \bar{y} = \frac{1}{s} \int_C y \delta ds,$$

where $s$ is the arclength. If the curve is $z = f(x)$, then $r = \langle x, f(x) \rangle$ and then $ds = |r'|dx = \sqrt{1 + f'(x)^2} dx$.

The second theorem of Pappus:

**Theorem 2.** The area of the surface of revolution equals the length of the curve times the perimeter of the circle for the centroid (with unit density).

We divide the curve $C$ into many segments. Then, each segment generates a slim annulus. The area of the annulus is $\Delta A \approx 2\pi x \Delta s \approx 2\pi x \sqrt{1 + f'(x)^2} \Delta x$.

Hence, the area is

$$A = \int 2\pi x \sqrt{1 + f'(x)^2} dx = \int_C 2\pi x ds = s(\frac{1}{s} \int_C x ds).$$
Moment of inertia

Suppose an object is rotating about line $L$ with angular speed $\omega$. What is the total kinetic energy? We know the kinetic energy for a moving object is $KE = \frac{1}{2}mv^2$. In our case here, different parts of the object have different speed. This formula is not applicable.

Now, imaging we divide the object into many small blocks. For one block, suppose the distance from the axis is $p$. Then, the kinetic energy is for the small block is $\Delta KE = \frac{1}{2} \Delta mv^2 = \frac{1}{2} \Delta m(\omega p)^2$. The total kinetic energy is therefore

$$KE = \frac{1}{2} \int \int \omega^2 p^2 dm = \frac{1}{2} I \omega^2.$$ 

The moment of inertia of a lamina about a straight line $L$ is defined to be

$$I = \int \int p^2 dm,$$

where $p$ is the perpendicular distance of the point to line $L$.

Suppose that the lamina is in $xy$ plane. $dm = \delta dA$. Then, about $z$-axis: polar moment of inertial: $p^2 = x^2 + y^2$ and we have

$$I_0 = \int \int (x^2 + y^2) \delta dA.$$ 

Again assume that the lamina is in the $xy$ plane: about $x$-axis: $I_x = \int \int y^2 \delta dA$. (If not in $xy$ plane, we should have $y^2 + z^2$). Similarly, $I_y$ is for the moment about $y$-axis.

Read the radius of gyration.

Example: Compute $I_x$ for the lamina bounded by $x = \pm y^4$ and $y = \pm 1$, with density $\delta(x, y) = x^2$.

$$I_x = \int_{-1}^{1} \int_{-y^4}^{y^4} y^2 (x^2) dx dy = \int_{-1}^{1} y^2 \frac{1}{3} 2y^{12} dy = \frac{2}{3} \frac{1}{15} \ast 2 = \frac{4}{45}.$$