14.1 Multiple integral: Double integral

For multiple integral, 2D → Double integral; 3D → triple integrals.

Let \( R \) be a region contained in the domain of the function \( f(x, y) \). The double integral is defined to be

\[
\iint_{R} f(x, y) dA = \lim_{|P| \to 0} \sum_{i} f(x^*_i, y^*_i) \Delta A_i,
\]

where \( P \) is a partition of the region \( R \) and \( A_i \) is the area of the \( i \)-th small region in the partition.

If \( f \geq 0 \), clearly \( f(x^*_i, y^*_i) \Delta A_i \) is the volume of the \( i \)-th cylinder. Hence, the double integral is the volume under the graph of \( z = f(x, y) \), above \( xy \) plane over region \( R \).

Rectangular region

Let \( R \) be a rectangle \([a, b] \times [c, d]\).

Draw a picture. Imaging we divide the rectangle into \( 3 \times 3 \) blocks. \( S = \sum_i \sum_j f(x_i, y_j) \Delta x \Delta y \). We can group each column first, or we can group each row first. Then, we have two ways to compute the final sum.

The double integral can be evaluated by the iterated integrals

\[
\iint_{R} f(x, y) dA = \int_{a}^{b} \int_{c}^{d} f(x, y) dy dx = \int_{c}^{d} \int_{a}^{b} f(x, y) dx dy.
\]

Example: Let \( f(x, y) = 4x^3 + 6xy^2 \) and \( R = [1, 3] \times [-2, 1] \). Evaluate the double integral \( \iint_{R} f(x, y) dA \) using two different iterated integrals.

Solution. The first:

\[
\int_{1}^{3} \int_{-2}^{1} (4x^3 + 6xy^2) dy dx = \int_{1}^{3} (4x^3 y + 2xy^3)|_{y=-2}^{y=1} dx = \int_{1}^{3} (12x^3 + 18x) dx = (3x^4 + 9x^2)|_{1}^{3} = 312.
\]

The second:

\[
\int_{-2}^{1} \int_{1}^{3} (4x^3 + 6xy^2) dx dy = \int_{-2}^{1} (x^4 + 3x^2 y^2)|_{x=1}^{x=3} dy = \int_{-2}^{1} (80 + 24y^2) dy = (80y + 8y^3)|_{y=-2}^{y=1} = 312.
\]
Explaination of the formula: If $f$ is nonnegative, the double integral is the volume under the surface and above the $xy$ plane over the region $R$. $A(x) = \int_c^d f(x, y)dy$ is the area of the cross section. If we integrate the area of the cross section again, we get the volume we want.

14.2 More general regions

If the region is not a rectangle, how do we compute the double integral? We use similar ideas as for the rectangle case. However, now, for a fixed $x$, the bounds of $y$ depend on $x$!

For vertically simple region $R$: for every vertical line, the intersection with the region is a single line segment (draw a picture).

$$a \leq x \leq b, \quad y_1(x) \leq y \leq y_2(x).$$

Similarly, we have horizontally simple region (draw a picture):

$$c \leq y \leq d, \quad x_1(y) \leq x \leq x_2(y).$$

Exercise: Recall that the region $x^2 + y^2 \leq 1$ can’t be written as $-1 \leq x \leq 1, -1 \leq y \leq 1$. Can you write it in a similar way like $\leq x \leq, \leq y \leq$?

$$-1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},$$

or

$$-1 \leq y \leq 1, -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}.$$

Example: Evaluate $\iint_R xy^2 dA$ in two ways where $R$ is the region bounded by $y = \sqrt{x}$ and $y = x^3$. $5/77$

Solution. The intersection of these two curves are $(0, 0)$ and $(1, 1)$.

By the picture if we express it in the vertically simple form, we have $0 \leq x \leq 1$ and $x^3 \leq y \leq \sqrt{x}$. This means we’ll integrate $y$ first:

$$\int_0^1 \int_{x^3}^{\sqrt{x}} xy^2 dy dx = \int_0^1 \frac{1}{3} xy^3 |_{y=x^3} dx = \int_0^1 \left( \frac{1}{3} x^{5/2} - \frac{1}{3} x^{10} \right) dx$$

$$= \left[ \frac{2}{7} x^{7/2} - \frac{1}{3} \frac{1}{11} x^{11} \right]|_0^1 = \frac{5}{77}. $$
If we express it in the horizontally simple form, we have $0 \leq y \leq 1, y^2 \leq x \leq y^{1/3}$.

$$\int_0^1 \int_{y^2}^{y^{1/3}} xy^2 \, dx \, dy = \int_0^1 \frac{1}{2} x^2 y^2 \bigg|_{x=y^2}^{x=y^{1/3}} \, dy = \int_0^1 \left( \frac{1}{2} y^{8/3} - \frac{1}{2} y^6 \right) \, dy$$

$$= \left( \frac{3}{11} y^{11/3} - \frac{1}{2} y^7 \right) \bigg|_0^1 = \frac{5}{117}.$$

Let’s look at more examples of double integrals on general regions to see more issues.

**Example:** Evaluate $\iint_R (6x + 2y^2) \, dA$ over the region bounded by the parabola $x = y^2$ and $x + y = 2$.

**Idea:** If we do $y$ first, then, we’ll result in two pieces of regions. If we do $x$ first, then our life is easier!

**Solution.** We evaluate $x$ first. The intersections are $(1, 1)$ and $(4, -2)$. Then, $-2 \leq y \leq 1, y^2 \leq x \leq 2 - y$. The integral is

$$\int_{-2}^1 \int_{y^2}^{2-y} (6x + 2y^2) \, dx \, dy = \int_{-2}^1 (3x^2 + 2xy^2) \bigg|_{x=y^2}^{x=2-y} \, dy$$

$$= \int_{-2}^1 (-5y^4 - 2y^3 + 7y^2 - 12y + 12) \, dy = \frac{99}{2}.$$

**Example:** Evaluate $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2) \, dx \, dy$.

**Idea:** We see that we are unable to find the antiderivative of $\sin(x^2)$. Then, we may write the region in another way. Note that it’s very wrong to have $\int_y^{\sqrt{\pi}} \int_0^{\sqrt{\pi}} \sin(x^2) \, dy \, dx$. After evaluating on $y$, the $y$ variable should disappear.

**Solution.** The region given is $0 \leq y \leq \sqrt{\pi}, y \leq x \leq \sqrt{\pi}$. In another way, $0 \leq x \leq \sqrt{\pi}, 0 \leq y \leq x$. Hence, the integral can be written as

$$\int_0^{\sqrt{\pi}} \int_0^x \sin(x^2) \, dx \, dy = \int_0^{\sqrt{\pi}} x \sin(x^2) \, dx = \int_0^{\sqrt{\pi}} \frac{1}{2} \sin(u) \, du = 1.$$

**Exercise:** Evaluate $\int_0^1 \int_{y^2}^{y^3} y(3x^2 + 1)^{1/3} \, dx \, dy$. 

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