Quiz 5

1. In this problem, you will set up integrals without evaluation.
(a). Set up iterated integrals for \( \int_S xy \, dS \) where \( S \) is the surface of revolution by revolving \( x = -y^2 + 3y - 2, x \geq 0 \) about \( x \)-axis.
We can use the modified cylindrical coordinates for \( yz \) to solve. In this sense, the surface becomes
\[
x = -r^2 + 3r - 2, \quad 1 \leq r \leq 2
\]
and \( y = r \cos \theta, \quad z = r \sin \theta \). The parametrization is then
\[
\vec{r}(r, \theta) = (-r^2 + 3r - 2, r \cos \theta, r \sin \theta), \quad 1 \leq r \leq 2, \quad 0 \leq \theta < 2\pi.
\]
Then, we have
\[
dS = |\vec{r}_r \times \vec{r}_\theta| \, drd\theta = \sqrt{r^2 + r^2(2r - 3)^2} \, drd\theta.
\]
The integral is therefore,
\[
\int_1^2 \int_0^{2\pi} (-r^2 + 3r - 2)(r \cos \theta) \sqrt{r^2 + r^2(2r - 3)^2} \, d\theta \, dr = 0
\]
(b). Set up \( \iint_S x \, dydz \) and \( \iint_S \vec{F} \cdot \vec{n} \, dS \) with \( \vec{F} = \langle x, y, z \rangle \). The surface \( S \) is the part of \( x + y = 1 \) inside \( x = z^2 \) and on the positive \( y \) side, with normal pointing in positive \( y \) direction.
Note that the first integral can be changed into
\[
\iint_S x \, dydz = \iint_S \langle x, 0, 0 \rangle \cdot \vec{n} \, dS
\]
Hence, both integrals can be treated in the same way.
The surface indeed appeared in Midterm 2. We can use \( x, z \) to parametrize.
\[
\vec{r} = \langle x, 1 - x, z \rangle.
\]
The region is the one bounded by \( x = 1 \) and \( x = z^2 \).
Then, \( \vec{n} \, dS = \pm \vec{r}_x \times \vec{r}_z \, dx \, dz \). I will omit the remaining steps.
Alternatively, you can use \( yz \) to parametrize: \( \vec{r} = \langle 1 - y, y, z \rangle \) and the region is the one bounded by \( 1 - y = z^2 \) and \( y = 0 \)
2. Here, I will let you solve the questions I mentioned in lecture.
(a) Let \( \mathbf{F} = \nabla \times (yz, -xz, z^2) \) and \( S \) is \( x^2 + y^2 = e^z \) between \( z = 0 \) and \( z = 2 \ln 2 \), with the normal point upward. We aim to compute
\[
\iint_S \mathbf{F} \cdot \mathbf{n} dS.
\]
In class, I explained how to apply Stokes backward so that we can evaluate line integrals to solve it. In this quiz, please solve this problem using surface independence. Explain why you can use surface independence. Since \( \mathbf{F} \) is the curl of another filed, we must have \( \nabla \cdot \mathbf{F} = 0 \). We can apply surface independence technique. Deforming the surface, we can use the two disks. One is \( S_1 : x^2 + y^2 \leq e^0 = 1 \) with normal pointing downward, while the other one is \( S_2 : x^2 + y^2 \leq e^{2 \ln 2} = 4 \) with normal pointing upward. Hence, we have:
\[
\iint_{S_2} \mathbf{F} \cdot \mathbf{k} dS - \iint_{S_1} \mathbf{F} \cdot \mathbf{k} dS
\]
We find \( \mathbf{F} = \langle x, y, -2z \rangle \).
\[
\iint_{S_1} \mathbf{F} \cdot \mathbf{k} dS = \iint_{S_1} (-2z) dS = \iint_{S_1} 0 dS = 0
\]
and
\[
\iint_{S_2} \mathbf{F} \cdot \mathbf{k} dS = \iint_{S_2} (-2z) dS = -4 \ln 2 \text{Area}(S_2) = -4 \ln 2 \cdot 4 \pi = -(16 \ln 2) \pi.
\]
The final answer is \(-(16 \ln 2) \pi - 0\)

(b) Suppose \( \mathbf{G} = \langle e^{x^2} + 3y + z, \sin(y^3) + x, 4y - 3z \rangle \). \( C \) is the intersection of \( z = x^2 + y^2 \) and \( z = 2x + 4y \) oriented clockwise when viewed from above. Compute the circulation of \( \mathbf{G} \) over \( C \).
Here, \( C \) is closed, and it is the boundary of the portion of the plane inside the paraboloid. Hence, we can set \( S \) be the part of the plane and the normal must point downward to satisfy the right hand rule. By Stokes
\[
\oint_C \mathbf{G} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{G}) \cdot \mathbf{n} dS
\]
Direct computation shows that \( \nabla \times \mathbf{G} = (4, 1, -2) \). The surface is the portion of the plane enclosed by \( C \) while \( C \) is determined by
\[x^2 + y^2 = 2x + 4y, \quad z = 2x + 4y.\]
Hence, $C$ is the intersection between a circular cylinder and a plane. It is an ellipse. The projection down in $xy$ plane is a circle centered at $(1, 2)$ with radius $\sqrt{5}$. This suggests that we can parametrize $S$ using $x, y$:

$$\vec{r} = (x, y, 2x + 4y), \ (x, y) \in D = \{(x, y) : (x - 1)^2 + (y - 2)^2 \leq 5\}.$$ 

$$\bar{n}dS = \pm \vec{r}_x \times \vec{r}_y dxdy$$

Since the normal is pointing down, we have

$$\bar{n}dS = \langle 2, 4, -1 \rangle dxdy$$

The integral is therefore:

$$\iint_D \langle 4, 1, -2 \rangle \cdot \langle 2, 4, -1 \rangle dxdy = 14 \text{Area}(D) = 14 \times 5\pi = 70\pi.$$ 

If the curve is the intersection between $4 = z^2 + x^2 + y^2$ and $z = 2x + 4y$, how will your solution approach be changed?

Applying Stokes, we can compute:

$$\iint_S \langle 4, 1, -2 \rangle \cdot \vec{n} dS$$

If we do again the parametrization using $x, y$, we will again get

$$14 \text{Area}(D)$$

However, now, the region $D$ is $x^2 + y^2 + (2x + 4y)^2 \leq 4$. This is a skewed ellipse. The area is hard to find. (If we know the major and minor axis, it would be easy). Hence, we may not do parametrization using $x, y$.

The key observation is that the curve $C$ is the intersection between the plane $z = 2x + 4y$ and the sphere. The plane passes through the center $(0, 0, 0)$. Hence, the curve $C$ is the great circle with radius 2. This means we may evaluate the integral directly. We check that

$$\vec{n} \parallel \text{normal of the plane} \Rightarrow \vec{n} = \frac{\langle 2, 4, -1 \rangle}{\sqrt{21}}$$

Hence, we have

$$\iint_S \langle 4, 1, -2 \rangle \cdot \frac{\langle 2, 4, -1 \rangle}{\sqrt{21}} dS = \frac{14}{\sqrt{21}} \iint_S dS = \frac{14}{\sqrt{21}} \text{Area}(S) = \frac{14}{\sqrt{21}} 4\pi.$$