**Math 212-Midterm 2**  
Fall 2017, Lei Li

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*Instructions: There are 6 problems with 100 points. Time is 75 minutes.*

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1. (10+10) 

(a). Find the value of the following iterated integral:

\[
\int_{0}^{2} \int_{y/2}^{1} ye^{x^3} \, dx \, dy.
\]

(b). Find the volume of the region bounded by \( z = e^{x^2-y^2} + x^2 \) and \( z = e^{x^2-y^2} - y^2 + 1 \) (Hint: Carefully identify the upper surface and use suitable coordinates to evaluate the double integral).

(a). \[
\begin{align*}
0 & \leq y \leq 2 \\
\frac{y}{2} & \leq x \leq 1
\end{align*}
\] 

\[
\Rightarrow \begin{cases}
0 \leq x \leq 1 \\
0 \leq y \leq 2x
\end{cases}
\]

\[
\int_{0}^{2} \int_{y/2}^{1} ye^{x^3} \, dx \, dy = \int_{0}^{1} \int_{0}^{2x} ye^{x^3} \, dy \, dx = \int_{0}^{1} 2x^2e^{x^3} \, dx
\]

\[
= \frac{2}{3} e^{x^3} \bigg|_{0}^{1} = \frac{2}{3} (e - 1)
\]

(b). \[
\begin{cases}
z = e^{x^2-y^2} + x^2 \\
z = e^{x^2-y^2} - y^2 + 1
\end{cases}
\Rightarrow \begin{cases}
z = e^{x^2-y^2} + x^2 \\
x^2 + y^2 = 1
\end{cases}
\]

This indicates that the projection of the solid onto xy plane is the inside of \( x^2 + y^2 = 1 \), or \( x^2 + y^2 \leq 1 \).
Above $R$, $e^{x^2-y^2} - y^2 + 1 \geq e^{x^2-y^2} + x^2$

So,

$$V = \iint_R \left( e^{x^2-y^2} + 1 - e^{x^2-y^2} - x^2 \right) \, dA$$

$$= \iint_R (1-x^2-y^2) \, dA.$$

In polar, $R$ is $0 \leq \theta < 2\pi$

$$0 \leq r \leq 1.$$

So

$$V = \int_0^{2\pi} \int_0^1 (1-r^2) \, rdr \, d\theta$$

$$= 2\pi \int_0^1 (r-r^3) \, dr = 2\pi \left( \frac{1}{2} - \frac{1}{4} \right)$$

$$= \frac{\pi}{2}$$
2. (15) Consider the region bounded by the parabolic cylinder \( x = z^2 \), \( zz \) plane and \( x + y = 1 \).

Suppose this region has a mass density \( \delta = x^2 \). Set up iterated integrals for the moment of inertia \( I_y \) with differentials \( dydzdx \).

The intersection between \( zz \) plane and \( x+y=1 \)

is \( x+1=1 \), or \( x=1 \).

So, the region in \( zz \) plane is determined by \( x = z^2 \) and \( x = 1 \).

In \( y \) direction, the solid is between \( zz \) plane and \( x+y=1 \).

We sketch it roughly in the shown picture.

\[
I_y = \iiint_T (x^2 + z^2) \delta \, dV
= \int_{-1}^{1} \int_{z^2}^{1-x} \int_{0}^{1-x} (x^2 + z^2) x^2 \, dy \, dx \, dz.
\]
3. (15) Choose ONE of the following to solve. Indicate clearly which one you choose.

(i). Let $R$ be the region bounded by $y - x = \sin(x + y)$ and the line segment $y = x$ $(0 \leq x \leq \pi/2)$. Evaluate the following double integral:

\[ \iint_R \sin(x + y) \, dA. \]

(ii). Consider the following iterated integral

\[ \int_1^2 \int_x^{x+\pi} \frac{\sin^2(y-x)}{x} \, dy \, dx. \]

Do suitable change of variables to transform the integration region into a rectangle and then evaluate this integral.

(i) We do change of variables.

\[ V = y-x, \quad U = x+y. \]

The segment $y=x$, $0 \leq x \leq \pi/2$, tells us that $0 \leq U \leq \pi$.

Then, in $U,V$ plane, the region is shown:

\[ J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = \frac{1}{2}. \]

So

\[ \iint_R \sin(x+y) \, dA = \iint_D \sin(U) \left| \frac{1}{2} \right| \, dU \, dV \]

\[ = \int_0^{\pi} \int_D \sin(U) \frac{1}{2} \, dv \, du = \frac{1}{2} \int_0^{\pi} \sin^2(U) \, du \]

Here, $\sin^2(U) = \frac{1}{2} (1 - \cos(2U))$. So $\int_0^{\pi} \sin^2(U) \, du = \frac{\pi}{2}$.
(ii) \[ 1 \leq x \leq 2 \]
\[ x \leq y \leq x + \pi \]

We set \( u = x \), \( v = y - x \). \( \Rightarrow \) \begin{align*}
\begin{cases}
  x = u \\
  y = u + v
\end{cases}
\end{align*}

Then \[ 1 \leq u \leq 2 \]
\[ 0 \leq v \leq \pi. \]

\[ J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1. \]

So \[ \int_1^2 \int_x^{x + \pi} \frac{\sin^2(y-x)}{x} \, dy \, dx = \int_1^2 \int_0^{\pi} \frac{\sin^2(v)}{u} \, |1| \, dv \, du \]

\[ = \ln 2 \cdot \int_0^{\pi} \sin^2 v \, dv = \frac{\pi}{2} \ln 2. \]

Again, \( \sin^2 v = \frac{1}{2} (1 - \cos(2v)) \).

\[ \int_0^{\pi} \sin^2 v \, dv = \frac{\pi}{2}. \]
4. (10+10)

(a) Evaluate the following integral using spherical coordinates:

\[ \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{18-z^2-y^2}}^{\sqrt{z^2+y^2}} (x^2 + y^2 + z^2) \, dz \, dx \, dy. \]

(b) Consider the right circular cone with base to be \( x^2 + y^2 \leq 1 \) in \( xy \) plane and vertex at \((0, 0, 2)\). Assuming the density is uniform, find its centroid using cylindrical coordinates.

(a) The region in \( xy \) is given by:

\[
\begin{align*}
0 \leq y &\leq 3 \\
0 \leq x &\leq \sqrt{9-y^2}
\end{align*}
\]

This is a quarter of the disk in the first quadrant.

For \( z \) direction: \( \sqrt{x^2+y^2} = z \) \( \Rightarrow \) \( z = r \) \( \Rightarrow \) \( \phi = \pi \)

\[ \sqrt{18-x^2-y^2} = z \] \( \Rightarrow \) \( x^2 + y^2 + z^2 = 18 \) \( \Rightarrow \) \( \rho = \frac{\sqrt{18}}{3 \sqrt{2}}. \)

Therefore the region is above \( \phi = \frac{\pi}{4} \), or the cone \( z = r \) but below the sphere \( \rho = 3 \sqrt{2} \).

We check that we compute the full solid in the first octant.

\[
\begin{align*}
z &= \sqrt{x^2+y^2} \\
x^2+y^2+z^2 &= 18 \\
&\Rightarrow z^2 = 9.
\end{align*}
\]

This means we get the full solid inside \( \rho = 3 \sqrt{2} \) but above \( \phi = \frac{\pi}{4} \) in 1st Octant.
So:
\[0 \leq \theta \leq \frac{\pi}{2},\]
\[0 \leq \phi \leq \frac{\pi}{4},\]
\[0 \leq \rho \leq 3\sqrt{2} .\]

\[
dz dxdy \rightarrow \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]
\[x^2 + y^2 + z^2 = \rho^2 \]

So
\[
\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{3\sqrt{2}} \rho^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{\pi}{2} \cdot \frac{1}{10} (3\sqrt{2})^5 \int_0^{\frac{\pi}{4}} \sin \phi \, d\phi
\]
\[= \frac{\pi}{10} (3\sqrt{2})^5 (1 - \frac{\sqrt{2}}{2})
\]

(b). The side surface is
\[
\frac{r}{1} + \frac{z}{2} = 1 .
\]

The region is
\[0 \leq \theta < 2\pi ,\]
\[0 \leq r \leq 1 ,\]
\[0 \leq z \leq 2(1-r) .\]
Let centroid be \((\bar{x}, \bar{y}, \bar{z})\)

By symmetry \(\bar{x} = \bar{y} = 0 .\)

\[
\bar{z} = \frac{1}{m} \iiint z \, dV = \frac{1}{Volume} \iiint z \, dV
\]

\[
= \frac{1}{2\pi \int_0^1 2(1-r^2) \, dr} \int_0^{2\pi} \int_1^{2(1-r)} \int_0^z z \cdot r \, dz \, dr \, d\theta
\]
\[= \frac{1}{2\pi \int_0^1 2(1-r^2) \, dr} \int_0^{2\pi} \int_1^{2(1-r)} 2\pi r(1-r) \, dr \]
\[= 6 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)
\]
\[= \frac{1}{2} .
\]
5. (7+8) Consider that \( f(x, y, z) = \rho \) (i.e. \( \rho = \sqrt{x^2 + y^2 + z^2} \)).

(a). Show that

\[
\nabla f = \frac{\vec{r}}{\rho},
\]

where \( \vec{r} = (x, y, z) \) is the position vector. In other words, we have \( \nabla \rho = \nabla f \).

(b). Assume that

\[
\vec{E} = \nabla f
\]

is an electric field (in physics, \( \phi = -f \) is called the static electric potential). Assume the charge density at \((1,0,0)\) is \( A \). What is the charge density at \((0,2,0)\)?

\[\text{(a)} \quad f_x = \frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} 2x = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\rho} \]

Similarly, \( f_y = \frac{y}{\rho}, \quad f_z = \frac{z}{\rho} \)

So \( \nabla f = \left( \frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho} \right) = \frac{1}{\rho} \vec{r} \). or \( \nabla \rho = \frac{\vec{r}}{\rho} = \hat{r} \).

\[\text{(b)} \quad \vec{E} = \frac{\vec{r}}{\rho}. \]

So \( \nabla \cdot \vec{E} = \nabla \cdot \left( \frac{\vec{r}}{\rho} \right) = \nabla \frac{\vec{r}}{\rho} + \vec{r} \cdot \nabla \left( \frac{1}{\rho} \right) \]

\[= \nabla \frac{\vec{r}}{\rho} + \vec{r} \cdot \left( -\frac{1}{\rho^2} \right) \nabla \rho \]

Since \( \nabla \rho = \frac{\vec{r}}{\rho} = \hat{r} \), we have

\[\nabla \cdot \vec{E} = \frac{\nabla \vec{r}}{\rho} - \frac{\vec{r}}{\rho^2} \cdot \frac{\vec{r}}{\rho} = \frac{3}{\rho} - \frac{1}{\rho} = \frac{2}{\rho}. \]

@ \((1,0,0)\) \( \nabla \cdot \vec{E}(1,0,0) = 2 \chie \frac{2}{A} = \frac{1}{\delta(1,0,0)} \)

@ \((0,2,0)\) \( \nabla \cdot \vec{E}(0,2,0) = 1 \)

So \( \frac{2}{A} = \frac{1}{\delta(0,2,0)} \)

\( \delta(0,2,0) = \frac{A}{2} \).
Alternatively, you may compute $\nabla \cdot \mathbf{E}$ directly:

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left\langle \frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho} \right\rangle$$

$$= \frac{\partial}{\partial x} \left( \frac{x}{\rho} \right) + \frac{\partial}{\partial y} \left( \frac{y}{\rho} \right) + \frac{\partial}{\partial z} \left( \frac{z}{\rho} \right)$$

$$\frac{\partial}{\partial x} \left( \frac{x}{\rho} \right) = \frac{1}{\rho} - \frac{x}{\rho^2} \frac{\partial \rho}{\partial x} = \frac{1}{\rho} - \frac{x}{\rho^2} \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{(x^2+y^2+z^2) - x^2}{(\sqrt{x^2+y^2+z^2})^3} = \frac{y^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}$$

Similarly, $\frac{\partial}{\partial y} \left( \frac{y}{\rho} \right) = \frac{x^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}$.

$$\frac{\partial}{\partial z} \left( \frac{z}{\rho} \right) = \frac{x^2+y^2}{(\sqrt{x^2+y^2+z^2})^3}.$$

You get the same $\nabla \cdot \mathbf{E}$. 

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(i). Let \( C \) be the curve as shown in the picture from \( A(0,0,0) \) to \( B(\pi,0,0) \). Compute the following line integral

\[
\int_C (\sin x + y) \, dx + (x + e^{\sin y}) \, dy + \sin(\cos z) \, dz.
\]

(ii). Let \( C \) be the curve given by

\[
\vec{r}(t) = (e^{5t}, \sqrt{(t-1)^8 + \ln(1+t)}, \arctan(t)), \quad t : 0 \to 1.
\]

Evaluate the following integral

\[
\int_C yz \, dx + xz \, dy + xy \, dz.
\]

(i). \( \int_C \vec{F} \cdot d\vec{r} \), \( \vec{F} = \langle \sin x + y, x + e^{\sin y}, \sin(\cos z) \rangle \).

\[
\nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\partial_x & \partial_y & \partial_z \\
\sin x + y & x + e^{\sin y} & \sin(\cos z)
\end{vmatrix} = \langle 0, 0, 0 \rangle
\]

\( \vec{F} \) is conservative. However, integrating \( e^{\sin y} \) and \( \sin(\cos z) \) is hard.
We then use path independence!

Let \( C \) be line segment from \( A \) to \( B \).

Then \( \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} \) by path independence.

\( \mathbf{\hat{r}}(t) = < \pi t, 0, 0 > \). \( t: 0 \rightarrow 1 \).

So \( \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left( \sin(\pi t) + 0 \right) \pi \cdot dt + (\ldots) 0 \cdot dt \\
+ (\ldots) 0 \cdot dt \\
= \int_0^1 \sin(\pi t) \pi \cdot dt = 2 \).

(b). \( \vec{F} = < yz, xz, xy > \). \( \nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
xz & yz & xz \\
yz & xz & xy
\end{vmatrix} = 0 \)

\( \vec{F} \) is conservative.

Actually, if we set \( f(x, y, z) = xyz \), we have \( \vec{F} = \nabla f = \nabla (xyz) \).

Then \( \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) \\
= x(1)y(1)z(1) - x(0)y(0)z(0) \\
= e^{\sqrt{2} + \pi} - 1 \cdot 0 \cdot 0 = e^{\sqrt{2} + \pi} \frac{\pi}{4} \).