Recall \( n \, dS = \langle dydz, dzdx, dxdy \rangle \).

Then, we have
\[
\iint_S P \, dydz + Q \, dzdx + R \, dxdy = \iint_S \langle P, Q, R \rangle \cdot n \, dS.
\]

We can see that the direction of normal \( n \) clearly matters. We must make sure that \( \vec{r}_u \times \vec{r}_v \) agrees with \( n \) (previously in lecture, we assume that \( n \) agrees with \( r_u \times r_v \) without mentioning, but from this lecture one, \( n \) will might be assigned and we must check).

\[
ndS = \pm \vec{r}_u \times \vec{r}_v \, dudv
\]

(\( dudv \) for double integrals is always positive.)

The flux of a vector field across a surface

Just like the line integrals, we can define the flux of \( F \) across \( S \) in the direction \( n \) to be
\[
\Phi = \iint_S F \cdot n \, dS = \iint_S F \cdot (r_u \times r_v) \, dudv,
\]

where we assume \( n \) is consistent with the parametrization (otherwise, we have a negative sign).

This measures for example the rate of flow of fluid going across \( S \). Clearly, the flux of a vector field (or the surface integral of a vector field) is equivalent to the surface integrals with respect to the coordinate elements

**Gauss’ laws:** If \( F \) is the gravitational field, then the flux of \( F \) across \( S \) is equal to \(-4\pi GM\) where \( M \) is the mass enclosed. If \( F \) is the electric field, then the flux is equal to \( Q/\epsilon_0 \) where \( Q \) is the total charge inside.

**Examples:**

1. Find the flux of \( F = \langle x, y, 3 \rangle \) out of the region \( T \) bounded by \( z = x^2 + y^2 \) and \( z = 4 \).
2. $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$ where $\mathbf{F} = \langle y, -x, z \rangle$ and $S$ is the surface $z = \theta$, $0 \leq \theta \leq \pi$ and $1 \leq x^2 + y^2 \leq 4$ with upward normal.

Parametrize the surface. $\mathbf{r} = \langle r \cos \theta, r \sin \theta, \theta \rangle$, $1 \leq r \leq 2, 0 \leq \theta \leq \pi$.

Then, $\mathbf{n}dS = \mathbf{r}_r \times \mathbf{r}_\theta drd\theta$

3. Compute the flux of $\mathbf{F} = \langle 2, 2, 3 \rangle$ across the surface $S$: $\mathbf{r}(u, v) = \langle u + v, u - v, uv \rangle, 0 \leq u, v \leq 1$, with the normal consistent with $\mathbf{r}_u \times \mathbf{r}_v$.

We compute directly that $\mathbf{n}dS = \mathbf{r}_u \times \mathbf{r}_v dudv = \ldots$

### 15.6 The Divergence Theorem

This is the generalization of the vector form of Green’s theorem to 3D space.

**Theorem 1.** Let $S$ be a closed surface in 3D space and the outer unit normal is $\mathbf{n}$. The region inside is $T$. Let $\mathbf{F}$ be continuously differentiable. Then,

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_T \nabla \cdot \mathbf{F} dV.$$

In the coordinates form, let $\mathbf{F} = \langle P, Q, R \rangle$ and the theorem is written as.

$$\iint_S Pdydz + Qdzdx + Rdxdy = \iiint_T (P_x + Q_y + R_z) dV.$$

**Example:** Recall this example we did in the surface integral section: Let $T$ be the solid bounded by $z = x^2 + y^2$ and $z = 4$. $\mathbf{F} = \langle x, y, 3 \rangle$. Compute the flux of $\mathbf{F}$ out of $T$.

**Physical meaning of divergence**

We have

$$\nabla \cdot \mathbf{F} = \lim_{{|V_i| \to 0}} \frac{1}{|V_i|} \iiint_{T_i} \nabla \cdot \mathbf{F} dV = \lim_{{|V_i| \to 0}} \frac{1}{|V_i|} \iint_{\partial T_i} \mathbf{F} \cdot \mathbf{n} dS$$

$\text{div}(\mathbf{F})$ is the ‘density’ of flux. It measures the rate that the material is diverging away from point $x$, or material taken away to generate the flow.

**Example:** Let $\mathbf{F} = \langle x + \cos y, y + \sin z, z + e^x \rangle$. Compute

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS$$
for the unit sphere where $n$ is the outer normal.

**Example:** Let $T$ be the cylinder with base radius $R$ and height $H$. $S = \partial T$ with outer normal. Compute

$$\int\int_S y \, dx \, dy \, dz$$

**Several applications (not required)**

Green’s formula If $\mathbf{F} = \nabla f$, then

$$\int\int_S \nabla f \cdot n \, dS = \int\int\int_T \Delta f \, dV$$

We can write the directional derivative $D_n f = \nabla f \cdot n$ to be $\partial f / \partial n$.

If $\mathbf{F} = f \nabla g$, then

$$\int\int_S f \nabla g \cdot n \, dS = \int\int\int_T \nabla \cdot (f \nabla g) \, dV = \int\int_T f \Delta g \, dV + \int\int_T \nabla f \cdot \nabla g \, dV$$

We do the same thing for $\mathbf{F} = g \nabla f$,

$$\int\int_S g \nabla f \cdot n \, dS = \int\int_T g \Delta f \, dV + \int\int_T \nabla f \cdot \nabla g \, dV$$

Taking the difference, we obtain the useful Green’s formula:

$$\int\int_S (f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n}) \, dS = \int\int\int_T (f \Delta g - g \Delta f) \, dV$$