15.4 Green’s theorem

Recall Green’s theorem:

**Theorem 1.** If $C$ is a simple closed curve, positively oriented (i.e. counterclockwisely oriented) and the region enclosed by it is $R$, then for any two continuously differentiable functions $P(x,y)$ and $Q(x,y)$, we have

$$\oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dA.$$ 

We can say

$\text{circulation} = \int \text{curl}.$

**Second version of Green’s theorem**

**Flux**

We start with the definition of outer flux.

Consider a flow of fluid in the plane with density $\delta$. The velocity field is $\mathbf{v}$. $C$ is a curve and $\mathbf{n}$ is the **unit normal** of $C$ such that when it is rotated counterclockwisely by $\pi/2$, we have $\mathbf{T}$. The net total mass of fluid going across the curve $C$ per unit of time is given by

$$\sum_i \delta_i \mathbf{v}_i \cdot \mathbf{n}_i \Delta s_i.$$ 

Hence, the **flux** of the fluid flow is

$$\Phi = \int_C \mathbf{F} \cdot d\mathbf{s}.$$ 

where $\mathbf{F} = \delta \mathbf{v} = \langle P, Q \rangle$. For a general vector field $\mathbf{F}$ where $\mathbf{F}$ does not necessarily have a physical meaning, the flux is just defined to be

$$\Phi = \int_C \mathbf{F} \cdot d\mathbf{s}.$$ 

Let’s figure out $\mathbf{n}$ in 2D: since $\mathbf{T} = \frac{1}{|\mathbf{r}'(t)|}(x'(t), y'(t))$, then

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \frac{1}{|\mathbf{r}'(t)|}(y'(t), -x'(t)) = \left( \frac{dy}{ds}, -\frac{dx}{ds} \right).$$
Since $ds = |r'(t)| dt$,

$$n ds = \langle y'(t), -x'(t) \rangle dt = \langle dy, -dx \rangle.$$  

The integral is then written as

$$\Phi = \int_C F \cdot n ds = \int_C P dy - Q dx.$$  

**Vector form of Green’s theorem**

Consider that $C$ is closed, positively oriented. With the convention of normal we choose, $n$ becomes the **outer normal** of the curve. Let $\tilde{P} = -Q$ and $\tilde{Q} = P$, we then have the following by the first version of Green’s theorem:

$$\oint_C F \cdot n ds = \iint_R \nabla \cdot F dA.$$  

is the vector form of Green’s theorem. It says that the flux is equal to the integration of divergence over the region inside.

**Example:** Compute the flux of $F = \langle 3xy^2 + 4x, 3x^2y - 4y \rangle$ across $C$ with normal pointing upward where $C$ is $y = \sqrt{4 - x^2}, y \geq 0$.

The idea is to construct another path so that the curve is closed. Then, we apply Green’s and take off the part we can compute easily.

**Physical meaning of divergence**

We apply the Green’s theorem on a circular disk:

$$\oint_C F \cdot n ds = \iint_R \nabla \cdot F dA.$$  

Since the integral of the divergence equals the flux, we may then call the divergence as **flux density**...

If we divide both sides by $\pi r^2$ and take $r \to 0$, we obtain the following formula:

$$\nabla \cdot F = \lim_{r \to 0} \frac{1}{\pi r^2} \oint_C F \cdot n ds.$$  

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We know the right hand side is the mass diverging away from the region inside $C$. In this sense, $\nabla \cdot F$ is therefore the net rate at which the fluid is diverging away from point $(x_0,y_0)$, or material taken away to generate the flow, as we talked in Section 15.1.

**Integration by parts for double integrals..(Will not test but it is interesting..)**

Q1. How do we integrate
\[
\iint_D \nabla \varphi dA?
\]
Here, $\nabla \varphi = \langle \varphi_x, \varphi_y \rangle$.

Q2. For integrals like
\[
\iint_D f \nabla \cdot F dA,
\]
can we do a certain type of integration by parts? What if we want to integrate
\[
\iint_D f \partial_x g dA?
\]

**15.5 Surface integrals**

The types of surface integrals are just like the types of line integrals.

**Surface integrals of a function over $S$**

Recall that in 14.8, we have studied that for $r(u,v)$, the surface area element is
\[
dS = |r_u \times r_v| dudv, \quad ndS = N dudv.
\]
(Draw the picture for surface element again.)

The surface integral of a function is defined to be the limit of Riemann sum again:

\[
\iint_S f(x,y,z) dS = \lim_{|\Delta S| \to 0} \sum_i f(x_i^*, y_i^*, z_i^*) \Delta S
\]
\[
= \lim_{|\Delta u| \to 0, |\Delta v| \to 0} \sum_i f(x_i^*, y_i^*, z_i^*) |r_u \times r_v| \Delta u \Delta v = \iint_D f(r(u,v)) |r_u \times r_v| dudv.
\]
\(|r_u \times r_v|\) is like the Jacobian, so this is very much like the change of variables for double integrals.

**Example:** Set up the integrals for the moment of inertial \(I_z\) of the hemisphere (the surface not the solid) \(x^2 + y^2 + z^2 = a^2, z \geq 0\) using both Cartesian parametrization and spherical parametrization. Assume density (mass per unit area) \(\delta = 1\).

**Surface integrals with respect to coordinate elements**

Let’s look at the definition of \(\int \int_S Rdxdy\).

Consider a small portion of the surface. The area is \(\Delta S\). Now, we don’t multiply \(\Delta S\) in the Riemann sum. Instead, we multiply the projection of this area onto \(xy\) plane. Just like we multiply \(\Delta x\) instead of \(\Delta s\) in line integrals:

\[
\int \int_S Rdxdy = \lim_{\Delta S \to 0} \sum_i f(x_i^*, y_i^*, z_i^*) \Delta A_{xy}
\]

To figure out the formulas to compute this, we recall the normal vector

\[
N = r_u \times r_v = \frac{\partial (y, z)}{\partial (u, v)} i + \frac{\partial (z, x)}{\partial (u, v)} j + \frac{\partial (x, y)}{\partial (u, v)} k.
\]

With the directional cosines, we can rewrite unit normal vector as

\[
\hat{n} = \frac{N}{|N|} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle, \quad \hat{n}dS = N dudv.
\]

Hence, \(\cos \gamma = \frac{1}{|N|} \frac{\partial (x, y)}{\partial (u, v)}\).

The angle between the surface portion and \(xy\) plane is equal to the angle between \(N\) and \(k\), which is \(\gamma\). We find that

\[
\Delta A_{xy} = \Delta S \cos \gamma = (|N| dudv) * \left( \frac{1}{|N|} \frac{\partial (x, y)}{\partial (u, v)} \right) = \frac{\partial (x, y)}{\partial (u, v)} dudv.
\]

We therefore find

\[
\int \int_S Rdxdy = \int \int_D R(x, y, z) \cos \gamma dS = \int \int_D R(x, y, z) \hat{k} \cdot \hat{n} dS
\]

Similarly, we have expressions for \(dydz, dzdx\).

\[
dxdy = \hat{k} \cdot \hat{n} dS = \frac{\partial (x, y)}{\partial (u, v)} dudv,
\]

\[
dydz = \hat{i} \cdot \hat{n} dS = \frac{\partial (y, z)}{\partial (u, v)} dudv,
\]

\[
dzdx = \hat{j} \cdot \hat{n} dS = \frac{\partial (z, x)}{\partial (u, v)} dudv.
\]
Hence, just like \( d\mathbf{r} = T\, ds = \langle dx, dy \rangle \) (or \( \langle dx, dy, dz \rangle \)) and \( \mathbf{n} ds = \langle dy, -dx \rangle \), we have

\[
\mathbf{n} dS = \mathbf{N} dudv = \langle dydz, dzdx, dxdy \rangle.
\]

Writing this type of surface integrals generally, we have

\[
\iint_{S} P\, dydz + Q\, dzdx + R\, dxdy = \iint_{S} \left(P\mathbf{i} \cdot \mathbf{n} + Q\mathbf{j} \cdot \mathbf{n} + R\mathbf{k} \cdot \mathbf{n}\right) dS
\]

\[
= \iint_{D} \left(P \frac{\partial (y, z)}{\partial (u, v)} + Q \frac{\partial (z, x)}{\partial (u, v)} + R \frac{\partial (x, y)}{\partial (u, v)}\right) dudv
\]

We usually don’t compute \( dydz, dzdx, dxdy \) and we compute \( \mathbf{N} dudv \) directly:

\[
\iint_{S} \langle P, Q, R \rangle \cdot \mathbf{n} dS = \iint_{S} \mathbf{F} \cdot \mathbf{n} dS = \iint_{S} \mathbf{F} \cdot \mathbf{N} dudv = \iint_{S} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dudv
\]

to compute this integral.

**Note that the direction matters for the surface integral.** \( \iint_{S} R\, dxdz = -\iint_{S} R\, dzdx \)

**Example:** Let \( S \) be the graph of \( z = h(x, y) \) for \( (x, y) \in D \), with upward normal. Reduce the surface integral

\[
\iint_{S} Q\, dzdx
\]

to a double integral over \( D. \)

\[
dzdx = \mathbf{j} \cdot \mathbf{n} dS = \mathbf{j} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dudv. \]

Then,

\[
\iint \langle 0, Q, 0 \rangle \cdot (\mathbf{r}_u \times \mathbf{r}_v) dudv
\]

Using \( x, y \) as parameters, \( \mathbf{n} dS = \langle -h_x, -h_y, 1 \rangle \, dxdy \) and \( \mathbf{n} \) is indeed pointing upward. Then, the integral is

\[
\iint_{D} Q(-h_y) \, dxdy
\]

\( dxdz \) could mean differently if you switch the normal of the surface even if we have the same surface \( z = h(x, y) \). Therefore, you have to use \( dzdx = \mathbf{j} \cdot \mathbf{n} dS \) to get the correct sign! Using the Jacobian is correct but switching the surface integral in \( dzdx \) into double integral in \( dxdz \) could again result in another negative sign (Just like the single integral \( \int f\, dx \) from 1 to 0, \( dx \) is negative. If we use double integral, \( dxdz \) always means the positive differential.)