14.9 (Part 1) Change of variables in double integrals

The double integral can be evaluated as

\[ \int \int_{R} f(x,y) \, dA = \int \int_{D} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv. \]

Fact:

\[ \frac{\partial(x,y)}{\partial(u,v)} \left( \frac{\partial(u,v)}{\partial(x,y)} \right|_{x=x(u,v), y=y(u,v)} \right) = 1 \]

The double integral can therefore also be computed using

\[ \int \int_{R} f(x,y) \, dA = \int \int_{D} f(x,y) \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|} \, du \, dv. \]

In the first way, we must write \( x, y \) in terms of \( u, v \). In the second way, we should have \( u, v \) in terms of \( x, y \). We should choose the more convenient one to use when solving problems.

**Example:** Find the area of the region bounded by \( y = 1/x \), \( x = 2/y \), \( y = 2x^2 \) and \( y = 4x^2 \).

**Solution.** We do change of variables: \( u = xy \) and \( v = y/x \). Then, the region becomes \( 1 \leq u \leq 2, 2 \leq v \leq 4 \), which is a rectangle in \( uv \) plane.

From here, \( y = u/x \) and hence \( v = u/(x^3) \) or \( x = (u/v)^{1/3} = u^{1/3}v^{-1/3} \).

\[ y = u^{2/3}v^{1/3}. \]

Then, the Jacobi is

\[ J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/3u^{-2/3}v^{-1/3} & -1/3u^{1/3}v^{-4/3} \\ 2/3u^{-1/3}v^{1/3} & 1/3u^{2/3}v^{-2/3} \end{vmatrix} = \frac{1}{9}v^{-1} + \frac{2}{9}v^{-1} = \frac{1}{3}v^{-1}. \]

Hence, the area is

\[ A = \int_{1}^{2} \int_{2}^{4} |J| \, dv \, du = \int_{1}^{2} \int_{2}^{4} \frac{1}{3v} \, dv \, du = \int_{1}^{2} \frac{1}{3} \ln(4/2) \, du = \frac{1}{3} \ln 2. \]

Another way to compute the Jacobian is to find

\[ \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -2y/x^3 & 1/x^2 \\ -2y/x^3 & 1/x^2 \end{vmatrix} = \frac{3y}{x^2} = 3v. \]

Then, the Jacobian we want is the inverse of it, or \( \frac{1}{3v} \). \( \square \)

**Example:** Evaluate the integral \( \int \int_{R} \frac{1}{(x^2+y^2)^2} \, dA \) where \( R \) is the one bounded by \( x^2 + y^2 = 6x, x^2 + y^2 = 2x, x^2 + y^2 = 8y, x^2 + y^2 = 2y \) using the change of variables \( u = 2x/(x^2+y^2) \) and \( v = 2y/(x^2+y^2) \).

We’ll do this using the inverse instead of solving out \( x, y \) explicitly.
10.2 Polar coordinates

Let $r$ be the distance of $(x, y)$ to the origin and $\theta$ be the angle measured from positive $x$-axis. Then, $(r, \theta)$ can describe a point in the plane uniquely as well.

To locate the point, we fist of all find the ray with angle $\theta$ and then use the distance $r$ to locate the point.

**Exercise:** Locate $(r, \theta) = (2, \pi/2)$ in the plane.

In general $r$ can be negative, and $\theta$ can be added with a multiple of $2\pi$. For example, $(2, 7\pi/6)$, $(2, -5\pi/6)$ and $(-2, \pi/6)$ will be the same point.

For convenience, we usually use $r \geq 0$. To cover the full plane once, we can do $r \geq 0$ and $0 \leq \theta < 2\pi$.

Clearly, we have the following relations if we use the convention $r \geq 0$:

$$x = r \cos \theta, \quad y = r \sin \theta.$$  

and

$$r = \sqrt{x^2 + y^2}, \quad \tan \theta = y/x \quad (x \neq 0)$$

**Example:** Find a pair of polar coordinates for $(x, y) = (-1, \sqrt{3})$.

$(r, \theta) = (2, 2\pi/3)$

**Example:** Find the polar equation for the circle centered at $(1/2, 0)$ with radius $1/2$.

**Solution.** The circle in Cartesian coordinates is $(x - 1/2)^2 + y^2 = (1/2)^2$ or $x^2 - x + y^2 = 0$. Plug in $x = r \cos \theta, y = r \sin \theta$, we have $r^2 - r \cos \theta = 0$, or $r = \cos \theta$. □

**Example:** Express the right half of the disk centered at $(x, y) = (0, 0)$ with radius 2 using polar coordinates.

By the picture, we see directly that $0 \leq r \leq 2, -\pi/2 \leq \theta \leq \pi/2$

**Example:** Write the disk $(x - 1/2)^2 + y^2 \leq 1/4$ in polar coordinates.

**Example:** Express $r = \sin(2\theta)$ in Cartesian coordinates.

We have $r = 2 \sin \theta \cos \theta$. Using $\sin \theta = y/r = y/\sqrt{x^2 + y^2}$ and $\cos \theta = x/r = x/\sqrt{x^2 + y^2}$, we have

$$\sqrt{x^2 + y^2} = 2 \frac{xy}{x^2 + y^2}.$$  

or $(x^2 + y^2)^3 = 4x^2y^2$. If we don’t include negative $r$, we should impose $xy \geq 0$. 

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14.4 Double integrals in polar coordinates

We consider the amplification factor for the transformation \( x = r \cos \theta \), \( y = r \sin \theta \). The Jacobian is

\[
J = \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta 
\end{vmatrix} = r.
\]

This tells us that \( dA = rdrd\theta \) (where \( dA \) is the area element in \( xy \) plane.)

By the picture (show this in class), we can determine that the area is \( dA = rdrd\theta \) directly. Hence, we can do \( dxdy \to rdrd\theta \).

Example: Evaluate the integral:

\[
\int_{0}^{2} \int_{\sqrt{4-x^2}}^{-\sqrt{4-x^2}} e^{-x^2-y^2} \, dy \, dx.
\]

Solution. The region is \( 0 \leq x \leq 2 \) and \(-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}\). This is the right half of the disk centered at \((0,0)\) with radius 2.

We use the polar coordinates. \( 0 \leq r \leq 2 \) and \(-\pi/2 \leq \theta \leq \pi/2 \). \( dydx \to rdrd\theta \). We have

\[
\int_{-\pi/2}^{\pi/2} \int_{0}^{r \sin \theta} e^{-r^2} r \, dr \, d\theta = \pi \int_{0}^{2} re^{-r^2} \, dr = -\frac{\pi}{2} e^{-r^2} \bigg|_{0}^{2} = \frac{\pi}{2} (1 - e^{-4}).
\]

Polar coordinates is convenient for radially simple region \( \alpha \leq \theta \leq \beta, r_1(\theta) \leq r \leq r_2(\theta) \).

Example: Let’s evaluate the volume of the solid bounded by \( z = x^2 + y^2 \) and \( z = y \).

Solution. Previously (see the lecture notes for volumes using double integrals), we agreed that the volume is

\[
V = \iint_{R} (y - x^2 - y^2) \, dA
\]

where \( R \) is the region bounded by the circle \( x^2 + y^2 = y \). This equation is \( r = \sin \theta \) in polar coordinates.

Letting \( r = 0 \), we have \( \theta = 0, \pi \) (note that \( 0 \to 2\pi \) will cover the disk twice.) hence, the region is \( 0 \leq \theta \leq \pi, 0 \leq r \leq \sin \theta \). The integral becomes

\[
\int_{0}^{\pi} \int_{0}^{\sin \theta} (r \sin \theta - r^2) r \, dr \, d\theta = \int_{0}^{\pi} \frac{1}{12} \sin^4 \theta d\theta.
\]
We are integrating even powers of $\sin \theta$. We do $\sin^2 \theta = (1 - \cos(2\theta))/2$. Then,
\[
\frac{1}{12} \int_0^\pi \frac{1}{4} [1 + \cos(2\theta)] d\theta,
\]
since the integral of $-2\cos 2\theta$ is zero. Lastly, $\cos^2(2\theta) = (1 + \cos(4\theta))/2$. The final answer is $\pi/32$.

Remark 1. We have a fact: if $f(x, y) = g(x)g(y)$, then $\int_a^b \int_a^b f(x, y) dA = (\int_a^b g(x) dx)^2$.

Example: Evaluate $I = \int_0^\infty e^{-x^2} dx$.

Solution. $I = \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-y^2} dy$. hence,
\[
I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_R e^{-x^2-y^2} dA,
\]
where $R$ is the first quadrant. In polar, $0 \leq r < \infty$ and $0 \leq \theta \leq \pi/2$. Then,
\[
I^2 = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta = \frac{\pi}{4}.
\]
hence, $I = \sqrt{\pi}/2$.

Exercise. Set up the integral for the volume under $f(x, y) = x^2$ and above the region $D = \{(x, y) : x^2 + y^2 \leq 4, x^2 + (y - 2)^2 \leq 4\}$ using polar coordinates.