Math 212-Lecture 1

Information

- Page: http://www.math.duke.edu/~leili/, then click ‘Teaching-Math 212’

- This course contains a lot and you may find the pace to be quite fast. If feel confused, spend extra time to understand as it is easy to fall behind with doubts and concerns.

- Office hours: 11:00am-12:00pm, Monday; 4:00-5:00pm Wed or by appointment. Physics 222. Help room: encourage you to go there. Academic Resource Center

- Sakai: Helpful materials for all Math 212 students on the shared site and grades on our own site.

- Homework assignments: finalized every Friday and collected every Tuesday

- There are some in-class or taken-home quizzes, to be announced.

- HW: 10%; Quiz: 5%; Mid-terms: 3 * 15%; Final 40%. Final exam: the four chapters will contribute roughly 20/20/20/40

12.1-Vectors in the plane

Def.

- Scalar: a single real number (pressure, speed)

- Vector: Both magnitude and direction (force, velocity). A directed line segment is used to represent a vector. Can move the vectors freely as long as the directed line segment carries the same magnitude and direction.

  - $\overrightarrow{QR}$ is a vector going from $Q$ to $R$.

  - For a point $P$ in space, we make a vector starting from the origin $O$ and ending at $P$. The vector $\overrightarrow{OP}$ describes the position of $P$, and is called the **position vector**.
When typing vectors denoted by a single letter, we sometimes use bold type \( \mathbf{a} \) or a letter with an arrow \( \vec{a} \). When we write by hand, we should use \( \vec{a} \).

**Vectors and ordered pairs of real numbers**

1. In a Cartesian plane:
   - \( \mathbf{i} \): a vector with magnitude 1 along \( x \)-axis;
   - \( \mathbf{j} \): a vector with magnitude 1 along \( y \)-axis.

   Any vector \( \mathbf{v} \) in the plane is expressed as \( \mathbf{v} = a \mathbf{i} + b \mathbf{j} \) for some real numbers \( a, b \). Then, the ordered pair \( \langle a, b \rangle \) represents uniquely a vector.

   Since \( \mathbf{i} = 1 \times \mathbf{i} + 0 \times \mathbf{j} \), then \( \mathbf{i} = \langle 1, 0 \rangle \). Similarly, \( \mathbf{j} = \langle 0, 1 \rangle \).

   Given \( Q(a_1, b_1), R(a_2, b_2) \), \( \overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \langle a_2 - a_1, b_2 - b_1 \rangle \).

2. \( \mathbf{v} = \langle a, b \rangle \). **Components**: \( a, b \); length or magnitude: \( \sqrt{a^2 + b^2} \) (Pythagorean theorem).

   Vectors in 2D plane: 2 components; Vectors in 3D space: 3 components.

   **Zero vector** is a vector with length zero, denoted as \( \mathbf{0} \). The direction is arbitrary.

3. **Unit vector**: A vector with length 1. If \( \mathbf{a} \neq \mathbf{0} \), \( \mathbf{u} = \frac{1}{|\mathbf{a}|} \mathbf{a} \) is a unit vector that shares the direction with \( \mathbf{a} \). We use \( \mathbf{u} \) to represent the direction of \( \mathbf{a} \). Sometimes denote it as \( \hat{a} \).

**Example**: Consider \( A(1, -2) \) and \( B(-1, -2 + 2\sqrt{3}) \). Let \( \mathbf{v} = \overrightarrow{AB} \). Find the magnitude and direction.

**Solution.** \( v = |\mathbf{v}| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{16} = 4 \).

\[ \hat{v} = \frac{1}{|\mathbf{v}|} \mathbf{v} = \frac{1}{2} \langle -1, \sqrt{3} \rangle. \]

**Basic operations**

1. \( \mathbf{u} = \langle u_1, u_2 \rangle, \mathbf{v} = \langle v_1, v_2 \rangle \). \( \mathbf{u} = \mathbf{v} \) iff \( u_1 = v_1, u_2 = v_2 \).

2. Addition and Scalar multiplication are performed componentwise.
3. Geometric meanings:

Addition: Triangle law or parallelogram law. Triangle rule indicates:
\[ \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \ldots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{A_1A_n} \]

Scalar multiplication: Scale the length. \( c > 0 \): the same direction; \( c < 0 \): opposite direction.

One useful identity:
\[ |c \langle v_1, v_2 \rangle| = |c| |v_1|^2 + c^2 |v_2|^2 = |c| \sqrt{v_1^2 + v_2^2} \]

Proof. \[ |c \langle v_1, v_2 \rangle| = |\langle cv_1, cv_2 \rangle| = \sqrt{c^2 v_1^2 + c^2 v_2^2} = |c| \sqrt{v_1^2 + v_2^2} \]

More examples

Example Let \( \mathbf{u} = 2i + 3j, \mathbf{v} = -2i + j \). Find \( 8\mathbf{u} - 4\mathbf{v} \).

Solution. \[ |8\mathbf{u} - 4\mathbf{v}| = 4|2\mathbf{u} - \mathbf{v}| \quad \mathbf{u} = \langle 2, 3 \rangle, \quad \mathbf{v} = \langle -2, 1 \rangle \quad 2\mathbf{u} - \mathbf{v} = \langle 4, 6 \rangle - \langle -2, 1 \rangle = \langle 6, 5 \rangle \]
Hence, \[ |2\mathbf{u} - \mathbf{v}| = \sqrt{6^2 + 5^2} = \sqrt{61} \] The final answer is \( 4\sqrt{61} \).

Read the example about relative velocity in book. We omit it here.

Example: An application In \( \triangle ABC \), \( M, N \) are the midpoints of \( AB \) and \( AC \). Show that \( |MN| = \frac{1}{2} |BC| \).

12-2: 3D vectors and dot product

Cartesian/Rectangular coordinates in space

- Three coordinate axes: \( x, y, z \). Right-handed. Note that if the directions \( \vec{a}, \vec{b}, \vec{c} \) are right-handed, then the directions \( \vec{b}, \vec{a}, \vec{c} \) are left-handed.

- Coordinate planes: \( xy, yz, xz \) planes. Octants.

- Any point \( P \) has a unique rectangular coordinate \( P(x, y, z) \).

- Given two points \( P_1(x_1, y_1, z_1) \) and \( P_2(x_2, y_2, z_2) \), the distance between them:
\[ d = |P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}. (Pythagorean again.) \]
Midpoint \( ((x_1 + x_2)/2, (y_1 + y_2)/2, (z_1 + z_2)/2) \).

Study the sphere yourself!
Vectors in space

Similar as the vectors in plane. Let $i, j, k$ be the basic unit vectors along the three axes. Then, $v = xi + yj + zk$. We identify

$$v = \langle x, y, z \rangle.$$

Length (magnitude), addition, scalar multiplication are similar as vectors in planes.

Dot product

$$a = a_1 i + a_2 j + a_3 k = \langle a_1, a_2, a_3 \rangle, \quad b = b_1 i + b_2 j + b_3 k = \langle b_1, b_2, b_3 \rangle.$$

The dot product is defined to be a number (scalar), given by

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Properties:

- $a \cdot a = |a|^2$
- $a \cdot b = b \cdot a$
- $a \cdot (b + c) = a \cdot b + a \cdot c$

We show the first as an example:

Proof. $a \cdot a = a_1 a_1 + a_2 a_2 + a_3 a_3 = (\sqrt{a_1^2 + a_2^2 + a_3^2})^2 = |a|^2$. □

Exercise: Show the distribution law holds.

Example Suppose $|a| = 2, |b| = 3$ and $a \cdot b = -4$. Compute $(a + 2b) \cdot (a - 2b)$ and $|a + 2b|$.

Solution. $(a + 2b) \cdot (a - 2b) = |a|^2 - 4|b|^2 = 4 - 4 \cdot 9 = -32. \quad |a + 2b| = \sqrt{(a + 2b) \cdot (a + 2b)}$ and distribute out the dot products. □