Quiz 2

1. Write out the region inside both \( x^2 + y^2 = 4 \) and \( (x - 2)^2 + y^2 = 4 \) but above \( y = \frac{\sqrt{3}}{3} x \) in polar coordinates.

In polar coordinates, \( y = \frac{\sqrt{3}}{3} x \) in the first quadrant is just \( \theta = \frac{\pi}{6} \). The first circle is \( r = 2 \) and the second circle is \( x^2 + y^2 = 4x \) or \( r = 4 \cos \theta \). The intersection of those two circles are at \( 2 = 4 \cos \theta \) or \( \theta = \pm \frac{\pi}{3} \).

Since we consider the region above \( \theta = \frac{\pi}{6} \), then, we must consider the part from \( \theta = \frac{\pi}{6} \) to \( \frac{\pi}{3} \) and the one from \( \frac{\pi}{3} \) to \( \frac{\pi}{2} \). By the picture, we find the region is the union of the following two

\[
\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}, \quad 0 \leq r \leq 2, \quad \text{together with} \quad \frac{\pi}{3} < \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 4 \cos \theta.
\]

2. Let \( R \) be the region bounded by \( y = x - 1, y = x - 2, \ x^2 - y^2 = 4, y^2 = x^2 - 9 \). (\( R \) is then in the first quadrant.) Evaluate the integral

\[
\int \int_R (x - y) \ln(x - y) dA.
\]

We do change of variables \( u = x - y \) and \( v = x^2 - y^2 \). Then, the region in the \( u, v \) plane becomes \( D : 1 \leq u \leq 2, \ 4 \leq v \leq 9 \). Then,

\[
J = \frac{\partial (u,v)}{\partial (x,y)} = \frac{1}{2x - 2y}.
\]

The integral is then

\[
\int \int_D (x-y) \ln(x-y)|\frac{1}{2(x-y)}| dudv = \frac{1}{2} \int \int_D \ln(x-y)dudv = \frac{1}{2} \int_1^2 \int_4^9 \ln u dudv = \frac{5}{2} \int_1^2 \ln u du.
\]

The last integral can be computed by integration by parts: \( \int \ln u du = u \ln u - \int 1du = u \ln u - u + C \). Then, the final answer is

\[
\frac{5}{2} (2 \ln 2 - 2 - 1 \ln 1 + 1) = 5 \ln 2 - \frac{5}{2}
\]