Math 212-Midterm 3
Fall 2015, Lei Li

Name: [Student Name]

Instructions: You have 50 minutes. There are 6 problems and 110 points in total. Besides, there are 10 bonus points. I will truncate your score to 100 if you get more. Use the big theorems to speed up your computation. Try your best.

<table>
<thead>
<tr>
<th>#</th>
<th>Total Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Problem 2</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Problem 3</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Problem 4</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Problem 5</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Problem 6</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Bonus</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
1. (15 points) Choose ONE of the following. Indicate clearly which one you choose.

- The area bounded by one loop of \( r = (\sin(2t), \sin(t)) \).
- \( \int_C \vec{F} \cdot d\vec{r} \) where \( \vec{F} = (e^{yz}, ze^{yz}, ye^{yz}) \) and \( C \) is \( \vec{r}(t) = (t^3, t^2, t), t : 0 \rightarrow 1 \)
- \( \int_C \vec{F} \cdot d\vec{r} \) where \( \vec{F} = (yz, xz, xy) \) and \( C \) is \( \vec{r}(t) = (te^t \cos t, t + t^2(t - 1), t \sin t), t : 0 \rightarrow 1 \).

- One loop is given as in the picture.

\[
A = \frac{\int_0^\pi \sin(2t) \cos t \, dt}{\int_0^\pi 2 \cos^2 t \sin t \, dt} = \frac{1}{3} 2u^2 \, du = \frac{4}{3}, \quad (u = \cos t)
\]

You may also use
\[
A = \int_C y \, dx = -\int_0^\pi \sin(t) \cdot 2 \cos(2t) \, dt = -\int_0^\pi 2(2u^3 - 1) \, du = \frac{4}{3}, \quad (u = \cos t)
\]

- \( \int_C \vec{F} \cdot d\vec{r} = \int_C e^{yz} \, dx + z e^{yz} \, dy + y e^{yz} \, dz \)

\[
= \int_0^1 t^3 e^{3t^2} \, dt + t e^{t^3} \int_0^1 2t \, dt + t^2 e^{t^3} \int_0^1 \, dt
\]

\[
= \int_0^1 6t^2 e^{t^3} \, dt = 2 \int_0^1 e^u \, du = 2(e - 1), \quad (u = t^3)
\]
\[ \vec{F} = \langle yz, xz, xy \rangle = \nabla(xy^2) \]

\( \vec{F} \) is conservative and \( \vec{F} = \nabla \phi \) where \( \phi = xy^2 \).

So \( \int_C \vec{F} \cdot d\vec{r} = \phi(F(1)) - \phi(F(0)) \)

\[ = \left[ xy^2 \right]_{t=1}^{t=0} - \left[ xy^2 \right]_{t=0}^{t=0} \]

\( F(0) = \langle 0, 0, 0 \rangle \)

\( F(1) = \langle e \cos 1, 1, \sin 1 \rangle \)

So \( \int_C \vec{F} \cdot d\vec{r} = e(\cos 1)(\sin 1) \)

\[ = \frac{1}{2} e \sin(2) \]
2. (25 points) Consider the region \( R \) bounded by \( y = x^2, \; y = 1 \) and \( y \)-axis. \( C \) is the boundary of the region oriented counterclockwise. Let \( \mathbf{v} = (P, Q) = (y^2, xy) \). Choose ONE of the following to evaluate, but for the one you choose, use two ways.

- The circulation 
  \[ \oint_C \mathbf{v} \cdot T \, ds \]
- The flux 
  \[ \iint_C \mathbf{v} \cdot n \, ds \]

- **Way 1:** 
  \[
  \oint_C \mathbf{V} \cdot T \, ds = \oint_P P \, dx + Q \, dy
  = \iint_R (Q_x - P_y) \, dA = \int_0^1 \int_{x^2}^1 (y - 2y) \, dy \, dx
  = -\frac{1}{2} \int_0^1 (1 - x^4) \, dx = -\frac{1}{2} \left( 1 - \frac{1}{5} \right) = -\frac{2}{5}
  
  **Way 2:** On \( C_1: \mathbf{F}(t) = \langle t, t^2 \rangle, \; 0 \leq t \leq 1 \)
  \[
  \oint_{C_1} \mathbf{V} \cdot T \, ds = \int_{C_1} y^2 \, dx + xy \, dy = \int_0^1 t^2 \, dt + t^3 \, 2 \, dt = \frac{3}{5}
  
  On \( C_2: \mathbf{F}(t) = \langle 1-t, 1 \rangle, \; 0 \leq t \leq 1 \)
  \[
  \oint_{C_2} \mathbf{V} \cdot T \, ds = \int_{C_2} y^2 \, dx + xy \, dy = \int_0^1 (-dt) + (1-t) \cdot 0 \, dt = -0.
  
  On \( C_3: \mathbf{F}(t) = \langle 0, 1-t \rangle, \; 0 \leq t \leq 1 \)
  \[
  \oint_{C_3} \mathbf{V} \cdot T \, ds = \int_{C_3} y^2 \, dx + xy \, dy = \int_0^1 0 \, dt + 0 \cdot (-dt) = 0.
  
  \]
Hence $\int_C \vec{V} \cdot \vec{n} ds = \frac{3}{5} - 1 + 0 = -\frac{2}{5}$.

• Flux. Way 1: $\int_C \vec{V} \cdot \vec{n} ds = \iint_R \nabla \cdot \vec{V} dA = \iint_R \left[ \frac{\partial (g^2)}{\partial x} + (xy)_y \right] dA$

  $= \int_0^1 \int_{x^2}^1 x \, dy \, dx = \int_0^1 x (1-x^2) \, dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

Way 2: On $C_1$: $F(t) = \langle t, t^2 \rangle$, $0 \leq t \leq 1$

  $\int_{C_1} \vec{V} \cdot \vec{n} ds = \int_{C_1} y^2 \, dy - xy \, dx = \int_0^1 t^4 \cdot 2t \, dt - t^3 \, dt = \frac{1}{3} - \frac{1}{4}$.

On $C_2$: $F(t) = \langle 1-t, 1 \rangle$, $0 \leq t \leq 1$

  $\int_{C_2} \vec{V} \cdot \vec{n} ds = \int_{C_2} y^2 \, dy - xy \, dx = \int_0^1 1 \cdot 0 \, dt - (1-t) \, (-dt) = \frac{1}{2}$.

On $C_3$: $F(t) = \langle 0, 1-t \rangle$, $0 \leq t \leq 1$

  $\int_{C_3} \vec{V} \cdot \vec{n} ds = \int_{C_3} y^2 \, dy - xy \, dx = \int_0^1 (1-t)^2 \cdot (-dt) - 0 \cdot dt = -\frac{1}{3}$

So $\int_C \vec{V} \cdot \vec{n} ds = \left(\frac{1}{3} - \frac{1}{4}\right) + \frac{1}{2} + \left(-\frac{1}{3}\right) = \frac{1}{4}$.
3. (15 points) Suppose \( \mathbf{v} = (x + x^2 + e^{y^2}, y + y^2 + \ln(x^2 + z^2 + 1), z + z^2 + \sin(xy)) \). \( S \) is the unit sphere and \( \mathbf{n} \) is the outer normal. Compute the outer flux

\[
\iint_S \mathbf{v} \cdot \mathbf{n} \, dS.
\]

(Hint: It’s possible to get the answer in 1 minute but the answer is not zero.)

The surface \( S \) is closed.

By divergence theorem

\[
\iiint_T \nabla \cdot \mathbf{v} \, dV
\]

where \( T \) is the unit ball and

\[
\nabla \cdot \mathbf{v} = (1+2x) + (1+2y) + (1+2z)
\]

so

\[
\iiint_T 3 + 2(x+y+z) \, dV = 3 \cdot \text{Vol}(T) = 3 \cdot \frac{4\pi}{3} = 4\pi.
\]

Here \( \iiint_T x \, dV = \iiint_T y \, dV = \iiint_T z \, dV = 0 \) by symmetry.
4. (25 points) Choose **ONE** of the following to evaluate. Indicate clearly which one you choose. (Hint: For both problems, you need \( \sin^2 \theta = (1 - \cos(2\theta))/2 \).)

- Let \( \vec{F} = \nabla \times \langle y, 0, x^2 z^2 \rangle \). Let \( S = \{(x, y, z) : x^2 + 4y^2 + 5z^2 = 9, z \geq -1\} \), and it is oriented such that the normal vector at \((0, 0, \sqrt{9/5})\) is pointing upward.

\[
\iint_S \vec{F} \cdot \vec{n} \, dS
\]

- Suppose \( C \) is the ellipse formed by the intersection of the cylinder \( x^2 + y^2 = 1 \) and the plane \( y + z = 0 \). \( C \) is oriented counterclockwise if we look down from above. Let \( \vec{v} = \langle e^{\cos z} + z^3/3, \ln(y^2 + 1) + y^2 z, \cos(z^2) + y^3/3 \rangle \). Compute

\[
\oint_C \vec{v} \cdot d\vec{r}.
\]

- Let \( \vec{G} = \langle y, 0, x^2 z^2 \rangle \).

\( C = \partial S \) is a curve in \( z = -1 \).

The equation is \( x^2 + 4y^2 + 5 = 9 \) or \( x^2 + 4y^2 = 4 \)

\( \vec{r}(t) = \langle 2\cos t, \sin t, -1 \rangle \quad 0 \leq t < 2\pi \).

The orientation of \( C \) is OK since it is counterclockwise viewed from above.

By **Stokes' Theorem**, \[
\iint_S \vec{F} \cdot \vec{n} \, dS = \oint_C \vec{G} \cdot d\vec{r} = \oint_C y \, dx + 0 \, dy + x^2 z^2 \, dz
\]

\( dz = 0 \cdot dt \quad \text{in} \quad z = -1 \quad \text{on} \quad C \).

So \[
\oint_C y \, dx = \int_0^{2\pi} \sin t \cdot (-2\sin t) \, dt = -2\pi.
\]

Some people use **Surface Independence**: \( \nabla \cdot \vec{F} = 0 \) since \( \text{div} (\text{curl}) = 0 \)

\[
\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_S \vec{F} \cdot \vec{k} \, dA \quad \text{Page 8} = \iint_S (-1) \, dA = -\text{Area}(S_1) = -2\pi.
\]

\( S_1 \) is the ellipse \( x^2 + 4y^2 = 4 \)

\( \{ z = -1 \} \).
Since \( \langle e^{\cos x}, \ln(y+1), \cos(z^2) \rangle \) is conservative, and \( C \) is closed, \( \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \langle \frac{\partial}{\partial x} y^2, \frac{\partial}{\partial y} \frac{y^3}{3}, \frac{\partial}{\partial z} \frac{z^3}{3} \rangle \cdot d\mathbf{r} \)

By Stokes's theorem,
\[
\iint_S \left[ \nabla \times \langle \frac{1}{2} z^3, y^2, \frac{1}{3} y^3 \rangle \right] \cdot \mathbf{n} \, dS = \iint_S \langle 0, z^2, 0 \rangle \cdot \mathbf{n} \, dS
\]

\( S \) is the part of \( y+z=0 \) inside \( x^2+y^2=1 \).

\( \mathbf{F}(x,y) = \langle x, y, -y \rangle \) on \( x^2+y^2 \leq 1 \).

So \( \mathbf{n} \, dS = \mathbf{F}_x \times \mathbf{F}_y \, dx \, dy = \langle -h_x, -h_y, 1 \rangle \, dx \, dy \), where \( h(x,y) = -y \).

\( \mathbf{n} \, dS = \langle 0, 1, 1 \rangle \, dx \, dy \).

Then \( \iint_D z^2 \, dx \, dy = \iint_D y^2 \, dx \, dy \) since \( z = -y \) on \( S \).

\[
\int_0^{2\pi} \int_0^1 r^2 \sin^2 \theta \, r \, dr \, d\theta = \frac{1}{4} \cdot \pi
\]

[Some people may parametrize \( C \) as \( \mathbf{F}(t) = \langle \cos t, \sin t, -\sin t \rangle \), \( 0 \leq t < 2\pi \).]

Computing \( \oint_C \frac{z^3}{3} \, dx + y^2 \, z \, dy + \frac{1}{3} y^3 \, dz \) is doable, but is not so interesting. ]
5. (15 points) Choose ONE of the following. Indicate clearly which one you choose.

- Consider \( \mathbf{v} = (P, Q) = \left( -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \) which satisfies \( P_y = Q_x \) except at the origin.
  Let \( C \) be the fancy curve that encloses the origin
  \[
  r(t) = (101 \cos(t)e^{\sin(2t)}, 101 \sin(t)(\cos(100t) - 1)^2), 0 \leq t < 2\pi,
  \]
  oriented counterclockwise. Evaluate
  \[
  \oint_C \mathbf{v} \cdot d\mathbf{r}.
  \]

- Consider the field \( F = (x^4 e^{\cos y}, x^4 e^{101x^{0.5} - 8z}, x^2 + y^2) \). Let \( S \) be the portion of the sphere \( x^2 + y^2 + (z-4)^2 = 17 \) above the \( xy \) plane, oriented such that the normal vector is the same as the outer normal of the sphere. Compute
  \[
  \iint_S F \cdot n \, dS.
  \]

- We use a small circle \( C_a \) with radius \( a \) to isolate the origin.
  Since \( P_y = Q_x \) except at the origin,
  \[
  \oint_C \mathbf{v} \cdot d\mathbf{r} = \oint_{C_a} \mathbf{v} \cdot d\mathbf{r} \quad \text{by Green's}
  \]

On \( C_a \): \( r(t) = \left( a \cos t, a \sin t \right), 0 \leq t < 2\pi \).
\[
\mathbf{v} = \left( -\frac{a \sin t}{a^2}, \frac{a \cos t}{a^2} \right) = \frac{1}{a} \left( -\sin t, \cos t \right)
\]
\[
d\mathbf{r} = \left( -a \sin t, a \cos t \right) dt
\]
So \( \mathbf{v} \cdot d\mathbf{r} = (\sin^2 t + \cos^2 t) dt = dt \)
\[
\oint_{C_a} \mathbf{v} \cdot d\mathbf{r} = \oint_0^{2\pi} dt = 2\pi ,
\]
\[ \nabla \cdot \mathbf{F} = 0. \]

\( \mathbf{F} \) satisfies the surface independence property.

\( \partial S = C \) is the curve in the xy plane, or
\[ x^2 + y^2 + 16 = 17 \text{ or } x^2 + y^2 = 1, \ z = 0 \]

So, if we set \( S_1 \) to be \( x^2 + y^2 \leq 1, \ z = 0 \) with an upward normal, then,
\[
\int \int_S \mathbf{F} \cdot \mathbf{n} dS = \int \int_{S_1} \mathbf{F} \cdot \mathbf{n} dS = \int \int_{S_1} (x^2 + y^2) dS
\]

In polar,
\[
\int_0^{2\pi} \int_0^1 r^2 r dr d\theta = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}.
\]
6. (15 points) Consider the region \( T \) that is the portion of the ball \( x^2 + (y - 1)^2 + z^2 \leq 1 \), below \( z = r \) but above \( z = -r \). Suppose the density is \( \delta(x, y, z) = 1 \). Write out, but do NOT evaluate, the integral for the moment of inertia about \( z \) axis.

(Hint: In the \( z \) direction, the lower bound sometimes is the cone and sometimes is the sphere, so the cylindrical coordinates are not suitable. Spherical coordinates are suitable but the upper bound for \( \rho \) is not a constant! Also be careful with the bounds for \( \theta \).)

\[
I_z = \iiint_T (x^2 + y^2) \, dV = \iiint_T (x^2 + y^2) \, dV \quad \text{since} \quad \delta = 1.
\]

In spherical, \( x^2 + (y-1)^2 + z^2 = 1 \) is \( \rho = 2 \sin \phi \sin \theta \).

- \( z = r \) is \( \phi = \frac{\pi}{4} \).
- \( z = -r \) is \( \phi = \frac{3\pi}{4} \).

So \( \frac{\pi}{4} \leq \phi \leq \frac{3\pi}{4} \).

The region is on the right of \( xz \) plane, so \( 0 \leq \theta \leq \pi \).

(One can set \( \rho = 0 \) or \( 2 \sin \phi \sin \theta = 0 \) to figure out \( \theta \))

\[
0 \leq \rho \leq 2 \sin \phi \sin \theta.
\]

\[
x^2 + y^2 = \rho^2 \sin^2 \phi.
\]

\[
dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.
\]

So \( \frac{\pi}{4} \leq \phi \leq \frac{3\pi}{4} \)

\[
\int_0^{\frac{\pi}{4}} \int_0^{\frac{3\pi}{4}} \int_0^{2 \sin \phi \sin \theta} \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta.
\]
Bonus (5+5 points)

- Suppose that \( \mathbf{F} = (P(x,y), Q(x,y)) \) is defined on region \( R \) which has no holes. If \( Q_x = P_y \), show that \( \mathbf{F} \) is conservative.

- Simplify \( \nabla \times (\mathbf{F} \times \mathbf{G}) \) by comparing it with \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \).

\[ \text{We show } \int_{C} \mathbf{F} \cdot d\mathbf{r} \text{ is path independent.} \]

Consider two curves \( C_1 \) and \( C \) who share endpoints.

\[ C - C_1 \] is a closed curve.

By Green's, \( \int_{C-C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C} P \, dx + Q \, dy = \iint_{R} (Q_x - P_y) \, dA = 0 \)

Where \( R \) is the region between \( C \) and \( C_1 \).

Hence \( \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \).

The integral \( \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{A} \mathbf{F} \cdot d\mathbf{r} \) is path independent.

Hence \( \mathbf{F} \) is conservative.
For $\nabla \times (F \times G)$, we should consider product rule and this property at the same time.

Hence,

$$\nabla \times (F \times G) = (\nabla \cdot G) F + (G \cdot \nabla) F$$

$$- (\nabla \cdot F) G - (F \cdot \nabla) G$$