Name: ________________________________

Instructions: There are 7 problems and a bonus problem. Your total score is at most 100. You have 50 minutes. No calculators are allowed.

<table>
<thead>
<tr>
<th>#</th>
<th>Total Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Problem 2</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Problem 3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Problem 4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Problem 5</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Problem 6</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Problem 7</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Bonus</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
1. \((5+5+5+5)\)

Suppose \(\vec{a} = \langle 1, -1, -1 \rangle\), \(\vec{b} = 2\vec{i} + \vec{j} + 2\vec{k}\). \(\vec{c}\) is another vector that satisfies \((\vec{a} \times \vec{b}) \cdot \vec{c} = -2\).

(a). Find the angle between \(\vec{a}\) and \(\vec{b}\).

(b). Find a vector \(\vec{a}_\perp\) that is perpendicular to \(\vec{b}\) such that \(\vec{a} - \vec{a}_\perp\) is parallel with \(\vec{b}\).

(c). Find a vector of length 2 that is perpendicular to both \(\vec{a}\) and \(\vec{b}\).

(d). Find the value of \(2\vec{a} \cdot (\vec{a} - 2\vec{b} + \vec{c}) \times (3\vec{a} - \vec{c})\).
2. (15) Suppose that the trajectory of a charged particle in a magnetic field is the helix \( \vec{r}(t) = (\cos t, \sin t, t) \). Find the plane that contains the line of intersection of \( x + y = 2z \) and \( 2x + z = 10 \), and that is parallel to the velocity of the particle at \( t = 0 \).
3. (2+2+1)

(a). Which of the following figures represents the surface \( z^2 - x^2 - y^2 = 1 \)?

(b). Which of the following figures represents the level curves of \( f(x, y) = y^2 - 2x^2 \)?

(c). Find an equation for the surface by revolving \( z = x^2, y = 0 \) about \( z \)-axis.
4. (10) Determine if the following functions are continuous at $(0, 0)$. If not, can we redefine $f(0, 0)$ to make them continuous at $(0, 0)$?

- $f(x, y) = \begin{cases} 
\frac{x^2 + 2y^2}{2x^2 + y^2}, & (x, y) \neq (0, 0), \\
1, & (x, y) = (0, 0).
\end{cases}$

- $f(x, y) = \begin{cases} 
\exp\left(-\frac{1}{x^2 + y^2}\right), & (x, y) \neq (0, 0), \\
1, & (x, y) = (0, 0).
\end{cases}$

$\exp(u)$ is the same as $e^u$. We use $\exp$ if the argument is a long expression.
5. (8+7) For the functions \( z = f(x, y) \) given by the expressions below, compute \( \frac{\partial z}{\partial x} \) at \((x, y) = (0, 0)\).

- \( z = xF(\cos(3x + 4y)) \). (You can assume that \( F(a) \) and \( F'(a) \) are known for any \( a \).)
- \( z = f(x, y) \) is determined by

\[
\sin(z + x) + \sin(z + y) + \sin(x + y) = 1.
\]

Suppose that \( f(0, 0) \) is between 0 and \( \pi/2 \) which you must find.
6. (15) Choose one to answer. Clearly indicate which one you choose.

- Let \( w = f(x, y) \) and \( x = r \cos \theta, y = r \sin \theta \). Regarding \( w \) as a function of \( r, \theta \), write \( \frac{\partial^2 w}{\partial r^2} \) in terms of \( f_{xx}, f_{xy}, f_{yy}, f_x, f_y, r \) and \( \theta \).

- Find a plane that is tangent to the 1-level set of \( F(x, y, z) = x^2 + y^2 - z \) and that is parallel with \( 5 + z = 2x - 2y \).
7. (8+12) There is a layer of sugar on the ground, which makes an ant quite happy. Suppose the concentration (mass per unit area) is given by

\[ f(x, y) = 2 + \arctan(x + y) + \ln(2 + \cos(2\pi xy)). \]

At \( t = 1 \) the ant is at \((0, 0)\) and it wants to head to higher concentration as fast as possible. The trajectory of the ant turns out to be

\[ \vec{r}(t) = \langle t^2 - 1, t^3 - 1 \rangle, \]

where \( t \) is the time.

(a). Suppose \( \ln(3) \approx 1.1 \). Get an estimate of the sugar concentration at \((x, y) = (0.1, 0.2)\) by choosing a suitable reference point and using linear approximation.

(b). How fast does the sugar concentrate change that the ant feels at \( t = 1 \)? Is the direction that the ant is crawling along the optimal? If not, tell it the optimal direction.
Consider the function
\[ f(x, y) = \begin{cases} 
\ln(1 + x^2 + y^2), & x > 0, \\
0, & x \leq 0.
\end{cases} \]

(a). Compute \( f_x(0, 0), f_y(0, 0) \) (If not exist, explain). Explain why \( f_x \) is not continuous at \((0, 0)\) (Hint: It is even not everywhere defined near \((0, 0)\)).

(b). Is this function continuous at \((0, 0)\)? Is this function differentiable at \((0, 0)\)?

(c). Use your answers in Part (a) and Part (b) to explain briefly the relationships among continuity of the function, existence of partial derivatives, differentiability, and continuity of partial derivatives for a function of two variables. (It is best to draw a diagram to illustrate the implications)