Math 212-Lecture 7

Cylinders and rulings

To define a cylinder, we need two things: 1. A curve $C$ in a plane. 2. A line $L$ that is not parallel to the plane.

The cylinder is the set of all lines that are parallel to $L$ and intersect with $C$. The lines are called rulings. Intuitively, we just shift the curve along the direction of $L$ and get a cylinder.

Surfaces with missing variables are cylinders.

Examples: Circular cylinder

$$x^2 + y^2 = 1.$$ 

We can obtain it by shifting the circle $C : x^2 + y^2 = 1, z = 0$ along $z$ axis since $z$ variable is missing in the equation.

Elliptic cylinder

$$4y^2 + 9z^2 = 36.$$ 

$x$ variable is missing. It is hence the cylinder by shifting $y^2/9 + z^2/4 = 1$ along $x$-axis.

Parabolic cylinder: $z = 4 - x^2$.

Revolution

Another class is the class of surfaces of revolution. We also need two things: 1. A curve $C$ in a plane. 2. A line in the same plane, called $L$.

We revolve the curve around $L$ and then get a surface. $L$ is called the axis of the surface of revolution.

The curve $f(x, y) = 0$ rotated about $x$-axis then generates

$$f(x, \sqrt{y^2 + z^2}) = 0.$$ 

If we rotate about $y$-axis, we have

$$f(\sqrt{x^2 + z^2}, y) = 0.$$ 

Comment: If we are going to revolve about $x$ axis. Then, we fix $x$ variable in the equation and replace the other one ($y$ or $z$) with $\sqrt{y^2 + z^2}$

Example: Sketch the graph of $z = x^2 + y^2$.

This is the surface of revolution by revolving $z = x^2, y = 0$ about $z$ axis.

Example: Describe the surface $x^2 + \sqrt{y^2 + z^2} = 4$. 

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Solution. Here, we see it can be obtained by revolving about $x$ axis as we have $\sqrt{y^2 + z^2}$.

The trace of the surface in $z = 0$ ($xy$ plane) is given by $x^2 + |y| = 4$, which is the union of two portions of parabolas. Hence, the surface is the surface of revolution by revolving the two portions of parabola $x^2 + y = 4, y \geq 0$ and $x^2 - y = 4, y < 0$ about $x$ axis.

□

Example:

$x = \sqrt{y^2 + z^2}$,

is the cone by revolving $x = |y|, z = 0$ about $x$-axis.

Quadric surfaces

The surfaces with equations of the form

$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + H = 0.$$  

Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$  

Consider $\frac{x^2}{1} + \frac{y^2}{1} + \frac{z^2}{1} = 1$. The general shape is like a football. Imagine stretching a sphere. To get a rough picture, we can sketch the traces in the $z = z_0$ planes.

We can shift the center as well. For example, if the center is shifted to $(1,1,2)$, then we have $\frac{(x-1)^2}{4} + \frac{(y-1)^2}{4} + (z-2)^2 = 1$.

Elliptic paraboloid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}.$$  

In the plane $z = z_0 > 0$, we have an ellipse. As $z$ grows larger, the ellipse is larger. In the $x = x_0$ plane, we have a parabola. The special paraboloid $z = x^2 + y^2$ can be obtained by revolving $z = x^2$ about $z$-axis.

Hyperbolic paraboloid:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}.$$  

In each $z = z_0$, we have a hyperbola.

Elliptic cone:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$  

In each $y = ky$ plane, the trace is just two lines. In the $z = z_0$ plane, the trace is an ellipse. This is a cone. The special cone $x^2 + y^2 = z^2$ can be obtained by revolving $z = \pm x$ about $z$ axis.
Hyperboloid of one sheet:

\[
x^2/a^2 + y^2/b^2 - z^2/c^2 = 1.
\]

In any \( z = z_0 \) plane, the trace is an ellipse. The whole surface has only one piece.

The surface \( y^2/4 + z^2 - x^2 = 1 \) is certainly also a hyperboloid of one sheet. The difference is that the axis is \( x \)-axis.

Hyperboloid of two sheets:

\[
z^2/c^2 - x^2/a^2 - y^2/b^2 = 1.
\]

If \( |z| < c \), there is no point. Only in \( z = z_0 > c \) or \( z = z_0 < -c \), the surface has a trace which is an ellipse. There are two pieces.