Math 212-Lecture 5

12-5: Curves and motion in space

Given a curve in space, how do we describe it? Imagine that a point is moving along the curve. Then, the point can trace the whole curve out. The coordinates of that point can be written as

\[ x = x(t), \quad y = y(t), \quad z = z(t). \]

In other words, the position vector of the point can be written as

\[ \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \]

which is a vector-valued function of \( t \). We can therefore use a vector-valued function \( \mathbf{r}(t) \) to describe a curve.

**Vector-valued functions: Differentiation, Integration**

We can simply look at each component since \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are independent of time \( t \).

For example, the derivative is defined to be

\[ \mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \langle x'(t), y'(t), z'(t) \rangle. \]

*Proof.*

\[ \mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [(x(t+\Delta t)i+y(t+\Delta t)j+z(t+\Delta t)k)-(x(t)i+y(t)j+z(t)k)] \]

\[ = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \mathbf{i} + \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} \mathbf{j} + \lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t} \mathbf{k} \]

\[ = x'(t)i + y'(t)j + z'(t)k. \]

\[ \square \]

Another example: the integration \( \int_a^b \mathbf{r}(t)dt \) is defined to be

\[ \int_a^b \mathbf{r}(t)dt = \lim_{\Delta t \to 0} \sum_{i=1}^{n} \mathbf{r}(t_i^*) \Delta t = (\int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt), \]

where we divide \([a, b]\) into \( n \) subintervals and \( t_i^* \) is a sample point from the \( i \)-th subinterval.
Exercise: Read P809 and understand why we can integrate each component to get the integral. If \( \mathbf{r} = u(t)\hat{a}(t) + v(t)\hat{b}(t) \), is it true or false that \( \int_0^1 \mathbf{r}(t)dt = \hat{a}(t) \int_0^1 u(t)dt + \hat{b}(t) \int_0^1 v(t)dt \)?

(The answer is false. Here, the two unit vectors are changing.)

Velocity and acceleration

If a particle is moving with position vector \( \mathbf{r}(t) \), then
\[
\mathbf{v}(t) = \mathbf{r}'(t) \\
\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).
\]

They are called the velocity and acceleration.

The speed: \( v(t) = |\mathbf{v}(t)| \). The scalar acceleration \( a(t) = |\mathbf{a}(t)| \).

Differentiation formulas

- \( (h(t)\mathbf{u}(t))' = h'(t)\mathbf{u}(t) + h(t)\mathbf{u}'(t) \)
- \( (\mathbf{u}(t) \cdot \mathbf{v}(t))' = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \)
- \( (\mathbf{u}(t) \times \mathbf{v}(t))' = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \)

Let’s show the second for vectors with 3 components.

Proof.

\[
(\mathbf{u}(t) \cdot \mathbf{v}(t))' = \left( \sum_{i=1}^{3} u_i(t)v_i(t) \right)' = \sum_{i=1}^{3} (u_i(t)v_i(t))'
\]

\[
= \sum_{i=1}^{3} (u'_i(t)v_i(t) + u_i(t)v'_i(t)) = \left( \sum_{i=1}^{3} u'_i(t)v_i(t) \right) + \left( \sum_{i=1}^{3} u_i(t)v'_i(t) \right)
\]

\[
= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t).
\]

Exercise: Explain in words why each step in the proof holds.

Two claims:
- If \( \mathbf{u}'(t) \perp \mathbf{u}(t) \), then \( |\mathbf{u}(t)| \) is a constant.
- \( |\mathbf{u}(t)|' = \mathbf{u}(t) \cdot \mathbf{u}'(t)/|\mathbf{u}(t)| \).
Example: A particle is moving on the unit sphere. Show that its velocity is perpendicular with the position vector.

Example: A charged particle in a magnetic field feels the Lorentz force $F = (qv) \times B$. Show that the charged particle in a magnetic field does not change the speed.

Solution.

$$ F \cdot v = 0 \Rightarrow a \cdot v = 0 \Rightarrow \frac{d}{dt}|v|^2 = 0 $$

Example: A moving particle has a given initial position $r(0) = 2i$ with initial velocity $v(0) = i - j$. Suppose the acceleration is $a(t) = 2i + 6tj$. Find the velocity and position at time $t$.

12.6: Arclength and curvature

Arclength

The curve $r(t) = (x(t), y(t), z(t))$ could be understood as the trajectory of a moving particle.

The distance the particle has traveled at $t$ is

$$ s = \int_0^t |v(\tau)|d\tau = \int_0^t \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}d\tau, $$

where $|v|$ is the speed. $s$ is called the arclength of the curve.

Clearly

$$ s'(t) = v(t) = |v(t)|. $$

$s$ is an increasing function with respect to $t$. Then, we can solve $t$ in terms of $s$: $t = t(s)$. Hence, the curve can be parametrized in terms of $s$:

$$ x = x(s), \ y = y(s), \ z = z(s). $$

Exercise: Use the chain rule to show $\frac{d}{ds}r(s) = 1$.

Example 1: Consider the helix $x(t) = a\cos(\omega t), y = a\sin(\omega t), z = bt$ where $a = 5, b = 12, \omega = 1$. Parametrize the helix in terms of the arclength.
Solution.

\[ s(t) = \int_0^t |r'(\tau)| d\tau = \int_0^t \sqrt{25 \sin^2 \tau + 25 \cos^2 \tau + 12^2} \, dt = 13t. \]

Hence, \( t = s/13 \) and

\[ r(s) = \langle 5 \cos(s/13), 5 \sin(s/13), 12s/13 \rangle. \]