12.4: Lines and planes in space

Suppose $P(x, y, z)$ is an arbitrary point on the geometric objects such as lines and surfaces. Goal: Find the equations for $x, y, z$ or equation for the position vector $\overrightarrow{OP}$.

Lines

Given a point on the line $P_0(x_0, y_0, z_0)$ and a vector $\mathbf{v} = (a, b, c)$ that is parallel with the line, we determine that any point on the line $P(x, y, z)$ satisfies the relation $\overrightarrow{P_0P} = tv$. The position vector of $P$ is given by

$$r = \overrightarrow{OP_0} + tv.$$  

This is called the vector equation of the line.

**Parametric equations**

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$  

**Symmetric equations (assume $a, b, c$ are all nonzero):**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$  

**Example:** (1). Find the parametric and symmetric equations of the line $L$ through $P_0(3, 1, -2)$ and $P_1(4, -1, 1)$. (2). Find a parametric equation for the line segment $P_0P_1$.

(For (2), find the range of $t$)

**Example:** Consider the skew lines $L_1: x = 1+t, y = -1-t, z = 2-3t$ and $L_2: x = t, y = t + 1, z = 2-t$. What is the distance between them?

**Exercise:** Suppose we have a line given by $\mathbf{r} = \mathbf{r}_0 + tv$. Find the distance from $Q(x, y, z)$ to the line.

Planes

Suppose $\mathbf{n} = (a, b, c)$ is perpendicular with Plane $\mathcal{P}$ (this is called normal vector) and $P_0(x_0, y_0, z_0)$ is on the plane. Then, $\overrightarrow{P_0P} \perp \mathbf{n}$. The vector equation:

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0.$$
where \( \vec{r} = \overrightarrow{OP} \) and \( \vec{r}_0 = \overrightarrow{OP}_0 \).

Scalar equation:

\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.
\]

Quick exercise: Consider the geometric object described by \( 2x - y + 3z = 6 \). It is a plane. Can you find out a normal vector for it? What does ‘6’ correspond to in the equation \( a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \)?

Angle between two planes

If two planes have normal vectors \( \vec{m} \) and \( \vec{n} \) respectively, the angle between them satisfies

\[
\cos \theta = \frac{\vec{m} \cdot \vec{n}}{||\vec{m}|| ||\vec{n}||}.
\]

Exercise: Draw a picture to illustrate that this formula is true.

Example: Suppose \( A(1, 0, -1) \), \( B(3, 3, 2) \), \( C(4, 5, -1) \) and \( D(0, 0, 1) \).

- Find the scalar equation for the plane \( ABC \)
- Find the distance from point \( D \) to the plane \( ABC \)
- Find the plane consisting of points that are equi-distant to \( A \) and \( B \).

Solution.

(1). A normal vector is \( \vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (-15, 9, 1) \). If we use the point \( A(1, 0, -1) \), then the equation will be \(-15(x - 1) + 9(y - 0) + 1(z + 1) = 0 \) or \(-15x + 9y + z + 16 = 0 \).

(2). \( d = ||\overrightarrow{AD} \cdot \vec{n}||/||\vec{n}|| \).

(3). A point on the point is the midpoint and the normal vector is \( \overrightarrow{AB} \)

Example: The two lines \( L_1 : \frac{x - 1}{2} = \frac{y - 3}{3} = z - 1 \) and \( L_2 : \frac{x - 1}{1} = \frac{y - 3}{2} = z - 1 \) intersect at a point.

- Find the scalar equation of the plane that contains these two lines.
- Find a plane that passes through \( A(-1, 1, 2) \) and \( B(-2, 3, 3) \) and is parallel with \( L_1 \).

Solution. Let \( \vec{v}_1 = (2, 3, 1) \) and \( \vec{v}_2 = (-1, 2, 1) \). A normal vector is \( \vec{n} = \vec{v}_1 \times \vec{v}_2 \). \( \square \)