Math 212-Lecture 32

13.5. Multivariable optimization: basic concepts and properties

Consider a region $R$ and a function $f$ defined on it.

- **Absolute maximum/absolute minimum** (also called global max/min): If the value at $(a,b)$ is bigger than or equal to the value at any other point in $R$, then $f(a,b)$ is called the global maximum over $R$. Similarly, we define the global minimum.

- **Local max/min**: We only compare the value at $(a,b)$ with values nearby. More clearly, $f$ achieves a local maximum at $(a,b)$ if there exists a small ball (or disk), called $B$, centered at $(a,b)$ with some radius $r > 0$ such that $f(a,b)$ is bigger than or equal to the value at any other point inside the intersection of $R$ and $B$.

- Maxima/Minima are called extreme values.

- A saddle point is a point that is a min along one direction but it is a max along another direction.

Pictures to show local max, local min and saddle.

**Lemma 1.** Any global extreme point is a local extreme point.

Existence of extreme values

**Theorem 1.** Suppose $R$ is a closed bounded region. If $f$ is continuous on the region $R$ (the function should also be continuous at the boundary), then the global maximum and global minimum exist, either in the interior or on the boundary.

**Example:** If $R = (-\infty, \infty)$, $f(x) = x$. The region is closed but it is unbounded. $f$ has no global max and min over it. If $R = (1,2]$, $f(x) = x$, though the region is bounded, $f$ has no global min on it since the region is not closed. If $R = [1,2]$, $f = x$, both global max and global min exist.
Interior points and critical points

Interior point means that you can find a small ball that contains the point and the whole ball is inside $R$.

If the local max/min happens in the interior, they must be critical point as we study below. Later, we’ll consider the max/min on a structure given by some level sets (constraints) where there are no interior points. If the max/min happens on the boundary of the region or the structures without interior points, we have to use other methods (Lagrange multiplier, reducing the number of variables etc).

c is called a critical point if $\nabla f(c) = 0$.

Lemma 2. If an interior point is a local extremum point, then it has to be a critical point.

- To prove this, you just look at the coordinate curves ($x$-curve and $y$-curve), since for a single-variable function at the critical point, the derivative is zero.
- If the local extremum is achieved on the boundary, then, $\nabla f$ does not have to be zero.
- A critical point may not be a local max or local min. It may be a saddle point.

Finding global extrema

We should consider all interior critical points, interior points where the derivatives do not exist and boundary points.

Example: $f(x, y) = xy - x - y + 3$. Find the global max and global min on the triangular region with vertices $A(0, 0), B(2, 0)$ and $C(0, 4)$.

Solution. 1. Find the critical points:

$\nabla f = 0$ or $y - 1 = 0, x - 1 = 0$. Point $(1, 1)$ is an interior point.

$f(1, 1) = 1 \times 1 - 1 - 1 + 3 = 2$.

2. Consider the boundary:

Boundary $AB$: $\alpha(x) = f(x, 0) = 3 - x, 0 \leq x \leq 2$. The two extrema $f(0, 0) = 3, f(2, 0) = 1$.

Boundary $AC$: $\beta(y) = f(0, y) = 3 - y, 0 \leq y \leq 4$. The two extrema $f(0, 0) = 3, f(0, 4) = -1$.

Boundary $BC$: The line is $y = -2x + 4$. Hence, $\gamma(x) = f(x, -2x + 4) = -2x^2 + 5x - 1, 0 \leq x \leq 2$. $\gamma'(x) = 0$ happens at $x = 5/4$. Hence,
\[ f(5/4, 3/2) = \gamma(5/4) = -25/8 + 25/4 - 1 = 17/8. \] \[ f(0, 4) \text{ and } f(2, 0) \text{ have been computed already.} \]

The global max is \( f(0, 0) = 3 \) and the global min is \( f(0, 4) = -1. \)

Sometimes, we know in advance that the highest point (global max) or lowest point (global min) exists in the interior, we can simply find them by finding the critical points for candidates.

**Example:** Find the global maximum point of \( f(x, y) = xy e^{-x^2 - y^2} \) in the first quadrant.

**Solution.** On \( x = 0 \) or \( y = 0 \), \( f = 0 \) and for \( x > 0, y > 0, f > 0. \) As \( (x, y) \to \infty, f \to 0 \), then there must a global max at some point \( x > 0, y > 0 \). That point has to be a critical point.

\[ \nabla f = 0. \quad f_x = ye^{-x^2 - y^2} - 2x^2 ye^{-x^2 - y^2} = 0 \quad \text{or} \quad y - 2x^2 y = 0. \]

Similarly, \( x - 2y^2 x = 0. \) Since \( x \neq 0, y \neq 0, \) then \( 1 - 2x^2 = 0 \) and \( 1 - 2y^2 = 0. \) There is only one candidate \( (1/\sqrt{2}, 1/\sqrt{2}). \) This has to be the global max. \( \square \)