Math 212-Lecture 26

Second version of Green’s theorem

Flux

We start with the definition of outer flux.

Consider a flow of fluid in the plane with density $\delta$. The velocity field is $v$. $C$ is a curve and $n$ is the unit normal of $C$ such that when it is rotated counterclockwisely by $\pi/2$, we have $T$. The net total mass of fluid going across the curve $C$ per unit of time is given by

$$\sum_i \delta_i v_i \cdot n_i \Delta s_i.$$ 

Hence, the flux of the fluid flow is

$$\Phi = \int_C F \cdot n \, ds.$$ 

where $F = \delta v = \langle P, Q \rangle$. For a general vector field $F$ where $F$ does not necessarily have a physical meaning, the flux is just defined to be

$$\Phi = \int_C F \cdot n \, ds.$$ 

Let’s figure out $n$ in 2D: since $T = \frac{1}{|r'(t)|} \langle x'(t), y'(t) \rangle$, then

$$n = T \times k = \frac{1}{|r'(t)|} \langle y'(t), -x'(t) \rangle = \langle \frac{dy}{ds}, -\frac{dx}{ds} \rangle.$$ 

Since $ds = |r'(t)| \, dt$,

$$n \, ds = \langle y'(t), -x'(t) \rangle \, dt = \langle dy, -dx \rangle.$$ 

The integral is then written as

$$\Phi = \int_C F \cdot n \, ds = \int_C P \, dy - Q \, dx.$$
Vector form of Green’s theorem

Let \( \tilde{P} = -Q \) and \( \tilde{Q} = P \), we then have the following by the first version of Green’s theorem:

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C P \, dy - Q \, dx = \oint_C \tilde{P} \, dx + \tilde{Q} \, dy \\
= \iint_R (\tilde{Q}_x - \tilde{P}_y) \, dA = \iint_R (P_x + Q_y) \, dA = \iint_R \nabla \cdot \mathbf{F} \, dA.
\]

is the vector form of Green’s theorem. It says that the flux is equal to the integration of divergence over the region inside.

**Example:** Compute the outer flux of \( \mathbf{F} = \langle 3xy^2 + 4x, 3x^2y - 4y \rangle \) across the \( C \) where \( C \) is \( y = \sqrt{4 - x^2}, y \geq 0 \).

The idea is to construct another path so that the curve is closed. Then, we apply Green’s and take off the part we can compute easily.

**Physical meaning of divergence**

We apply the Green’s theorem on a circular disk:

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_B \nabla \cdot \mathbf{F} \, dA.
\]

Since the integral of the divergence equals the flux, we may then call the divergence as **flux density**...

If we divide both sides by \( \pi r^2 \) and take \( r \to 0 \), we obtain the following formula:

\[
\nabla \cdot \mathbf{F} = \lim_{r \to 0} \frac{1}{\pi r^2} \oint_C \mathbf{F} \cdot \mathbf{n} \, ds
\]

We know the right hand side is the mass diverging away from the region inside \( C \). In this sense, \( \nabla \cdot \mathbf{F} \) is therefore the net rate at which the fluid is diverging away from point \( (x_0, y_0) \), or material taken away to generate the flow, as we talked in Section 15.1.

**Integration by parts for double integrals..(Will not test but it is interesting..)**

Q1. How do we integrate

\[
\iint_D \nabla \varphi \, dA?
\]
Here, $\nabla \varphi = \langle \varphi_x, \varphi_y \rangle$.

Q2. For integrals like

$$\iint_D f \nabla \cdot F \, dA,$$

can we do a certain type of integration by parts? What if we want to integrate

$$\iint_D f \partial_x g \, dA?$$