Math 212-Lecture 23

15.2 Continue...

Line integrals and vector fields

Suppose \( \mathbf{F} = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k} \) is the vector field. The oriented curve \( C \) is given by \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) is a curve.

\[ T = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \]

is the unit tangent vector.

**Line integral of the vector field** or the work as some people call it is defined to be the limit of Riemann sum again

\[ W = \int_C \mathbf{F} \cdot \mathbf{d}s = \lim_{\Delta s_i \to 0} \sum_i \mathbf{F}(x^*_i, y^*_i, z^*_i) \cdot T \Delta s_i. \]

\( \mathbf{F} \cdot \mathbf{T} \) is the projection of the force onto the direction of displacement. Physically, if you dot the force with the displacement, you get the work.

\( ds = |\mathbf{r}'(t)|dt \) and \( \mathbf{T} ds = \mathbf{r}'(t) dt = d\mathbf{r} \) is the displacement. \( T \) is therefore the direction of the infinitesimal displacement and \( ds \) is the magnitude \( d\mathbf{r} \). Hence,

\[ d\mathbf{r} = (dx, dy, dz) = \langle x'(t), y'(t), z'(t) \rangle dt, \]

\[ ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt \]

Having these,

\[ W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \langle P, Q, R \rangle \cdot d\langle x, y, z \rangle = \int_C Pdx + Qdy + Rdz. \]

The orientation matters for the work because \( T \) will be changed if the orientation changes.

Hence, the work can be written in both types of integrals.

\[ \int_C Pdx + Qdy + Rdz, \int_C \mathbf{F} \cdot d\mathbf{r}, \int_C \mathbf{F} \cdot T ds. \text{ They are all equivalent.} \]

If you see any one of them, you should recall the other two.

**Example:** Consider the force \( \mathbf{F} = \langle z, x, -y \rangle \). Let \( A(0, 0, 0) \) and \( B(1, 1, 1) \).

Suppose \( C_1 \) is the line segment from \( A \) to \( B \) and \( C_2 \) is \( \mathbf{r}(t) = \langle t, t^2, t^3 \rangle, 0 \leq t \leq 1 \). These two curves have the same endpoints. Compute the work along \( C_1 \), denoted by \( W_1 \) and the work along \( C_2 \), denoted by \( W_2 \). Is \( W_1 = W_2 \)?
Solution. For $C_1$, parametrize the curve as $r = \langle 0, 0, 0 \rangle + t(1-0, 1-0, 1-0)$, $0 \leq t \leq 1$. As long as we have this, the remaining part is easy:

$$W_1 = \int_{C_1} zdx + xdy - ydz = \int_0^1 tdt + tdt - tdt = \frac{1}{2}.$$

For $C_2$, we can compute directly:

$$W_2 = \int_{C_2} zdx + xdy - ydz = \int_0^1 t^3 dt + t^2 dt - t^2 dt = \frac{1}{4} + \frac{2}{3} - \frac{3}{5} = \frac{19}{60}.$$

Hence, $W_1 \neq W_2$.

In this example, the work depends on the concrete path, not just on the endpoints.

Questions: Are there any fields for which the line integral only depends on endpoints? This will be answered in the next section...

- Compute the line integral of $\mathbf{F} = \langle 3y, -2x \rangle$ over the curve $y = x^2$ for $0 \leq y \leq 1$ oriented from right to left.
- Let $\mathbf{r} = \langle t^3, t^2, t \rangle, 0 \leq t \leq 1$. Compute $\int_C \mathbf{F} \cdot T ds$ where $\mathbf{F} = \langle e^{yz}, 0, ye^{yz} \rangle$.

15.3 The fundamental theorem and independence of path

The fundamental theorem

**Theorem 1.** Suppose $f$ is differentiable and $C$ is some curve from $A$ to $B$, then

$$\int_C \nabla f \cdot dr = f(B) - f(A).$$

This theorem says that the line integral of gradient vector field is independent of how the path goes from $A$ to $B$.

Intuitively, $\nabla f \cdot dr = f_x dx + f_y dy + f_z dz = df$ which is the differential of $f$ and hence

$$\int_C \nabla f \cdot dr = \int_a^b df = f(r(b)) - f(r(a)).$$

This process can also be written in the parametric form:

$$\int_a^b \langle f_x, f_y, f_z \rangle \cdot \langle x', y', z' \rangle dt = \int_a^b \frac{df}{dt} dt = f(x(t), y(t), z(t)) \big|_a^b.$$
In last lecture, $F = (z, x, -y)$. We have seen two curves from $(0, 0, 0)$ to $(1, 1, 1)$ but the works on them are different. This means that $F$ can’t be a gradient vector field. Therefore... we have the issue of path independence...

Again, previously I told you that $\int_C ydx \neq xy|_A^B$. After we have studied this, we can see that actually, $\int_C ydx + xdy = xy|_A^B$.

Are there other fields besides gradient fields for which the line integral only depends on endpoints? The answer will be no...

Independence of path

The integral $\int_C F \cdot dr = \int_C F \cdot Tds$ is said to be independent of the path if the integral only depends on the endpoints $A, B$, not on the particular choice of the path joining them. In this case, we may write

$$\int_C F \cdot Tds = \int_A^B F \cdot Tds$$

Theorem 2. The line integral is independent of path if and only if

$$F = \nabla f$$

for some scalar function $f$.

In this case, $F$ is called a conservative field and $f$ is called the potential function of $f$.

In physics, the potential is defined to be $F = -\nabla f$.

By the fundamental theorem, that the latter implies the former is clear. We are not going to prove that the left implies the right. Read the book for the proof.

Theorem 3. Suppose region $D$ in $xy$ plane has no holes, then vector field $F = (P, Q)$ is conservative if and only if $P_y = Q_x$.

The above says $F = \nabla f$ if and only if $P_y = Q_x$. For $(\Rightarrow)$ direction, it is exactly the Clairaut’s theorem. Later, we’ll also use the Green’s theorem to show the reverse direction.

How about 3D case? For $F = \nabla f$, clearly it also should be consistent with the Clairaut’s theorem somehow. The Clairant’s theorem for a function of three variables can be written concisely as

$$\nabla \times \nabla f = 0.$$
Theorem 4. In 3D case, suppose $D$ is some region that has no holes. $F$ is conservative if and only if
\[ \nabla \times F = 0. \]

For $\Rightarrow$ direction, it is $\nabla \times \nabla f = 0$. For the $\Leftarrow$ direction, we’ll use the Stokes’ theorem to show.