14.8 Surface Area

Previously, we see that a vector valued function with a single variable \( r(t) \) is a curve in space.

Now, if the function is vector-valued but has two variables (parameters)

\[
\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle,
\]

where \( \mathbf{r} \) again is the position vector of the point, what will the object be?

Fixing \( v = v_0 \), \( \mathbf{r}(u,v_0) \) is a curve. Now, for different \( v = v_1 \), it’s another curve. The object is thus a family of curves, and they form a surface.

Parametrization of a surface

By ‘parametrizing’, we mean we find two parameters \( u, v \) and we write the position vector \( \mathbf{r} = \langle x, y, z \rangle \) of a point on the surface in terms of \( u, v \). When \( u, v \) range over its domain, the terminal point of \( \mathbf{r} \) then sweeps the whole surface.

**Example:** The graph \( z = f(x,y) \) is a surface in 3D space. Parametrize it.

\[
\mathbf{r} = \langle x, y, z \rangle. \quad \text{We choose } x, y \text{ as the parameters. Then, } \mathbf{r} = \langle x, y, f(x,y) \rangle.
\]

**Example:** Let \( \rho, \phi, \theta \) be the spherical coordinates. The function \( \rho = h(\phi, \theta) \) gives a surface in the space. (Example is \( \rho = 2 \).) Parametrize this surface.

**Solution.** \( \mathbf{r} = \langle x, y, z \rangle = \langle h(\phi, \theta) \sin \phi \cos \theta, h(\phi, \theta) \sin \phi \sin \theta, h(\phi, \theta) \cos \phi \rangle. \)

**Example:** Parametrize the rectangle \( 0 \leq x \leq 2, 0 \leq y \leq 3, z = 1 \).

**Solution.** This is a special case of the first example, \( z = f(x,y) = 1 \) and hence

\[
\mathbf{r}(x,y) = \langle x, y, 1 \rangle, \quad 0 \leq x \leq 2, 0 \leq y \leq 3
\]
Computing the area of a surface

Given a parametric surface
\[ r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k = (x(u,v), y(u,v), z(u,v)), \]
we call it smooth if
\[ r_u = (x_u, y_u, z_u), \quad r_v = (x_v, y_v, z_v), \]
are both nonzero and nonparallel.

Consider the small area for the rectangle \( \Delta u \Delta v \) in \( u-v \) plane. It’s a parallelogram on the surface under the mapping \( r(u,v) \). Draw a picture.

One edge is \( a = r(u + \Delta u, v) - r(u, v) \approx r_u \Delta u \). The other edge is similarly \( b \approx r_v \Delta v \). The area is therefore
\[ \Delta S \approx |a \times b| \approx |r_u \times r_v| \Delta u \Delta v. \]
\( N = r_u \times r_v \) is a normal vector of the surface. The total area is
\[ A = a(S) = \iint_{u,v} dS = \iint_{u,v} |r_u \times r_v| dudv. \]
\[ dS = |r_u \times r_v| dudv \]
is the surface area element. Let \( n = N / |N| \). \( d\vec{S} = ndS = N dudv = r_u \times r_v dudv \) is the directed surface area element.

Here, we see that \( |r_u \times r_v| \) plays the same role as the Jacobian in the change of variables for double integrals. It’s the amplification factor between the areas.

**Example:** If \( u, v \) are the Cartesian coordinates \( x, y \), then \( r = \langle x, y, f(x,y) \rangle \).

It is the graph of \( z = f(x,y) \). \( r_x \times r_y = \langle -f_x, -f_y, 1 \rangle \).

This makes sense as it is just \( \nabla F \) where \( F = z - f(x,y) \). The area is
\[ A = \iint_R \sqrt{1 + f_x^2 + f_y^2} dxdy. \]

We compute the area of the ellipse cut from \( z = 2x + 2y + 1 \) by \( x^2 + y^2 = 1 \).

What if the surface is the one cut from \( x = 2y + 2z + 1 \) from \( y+z = 1, y = 0, z = 0 \)?

**Example:** Find the area of the spiral ramp \( z = \theta, 0 \leq r \leq 1, 0 \leq \theta \leq \pi \).
Solution. We parametrize the surface \( r(r, \theta) = \langle r \cos \theta, r \sin \theta, \theta \rangle \).

\[
r_r \times r_\theta = \langle \cos \theta, \sin \theta, 0 \rangle \times \langle -r \sin \theta, r \cos \theta, 1 \rangle \]

The magnitude of this is \( \sqrt{1 + r^2} \)

The integral is

\[
\int_0^1 \int_0^\pi \sqrt{1 + r^2} d\theta dr = \pi \int_0^{\pi/4} \sec^3 \theta d\theta = \frac{\pi}{2} (\sqrt{2} + \ln (1 + \sqrt{2}))
\]

Example. Compute the area of the portion of \( z^2 = 3(x^2 + y^2) \) below \( z = 3 \), and above \( xy \) plane.

\( r = \langle r \cos \theta, r \sin \theta, \sqrt{3}r \rangle \). \( 0 \leq r \leq \sqrt{3}, 0 \leq \theta < 2\pi \). The magnitude of \( r_r \times r_\theta \) is \( 2r \). Then, \( \int_0^{\sqrt{3}} \int_0^{2\pi} 2rd\theta dr \).