14.9 (Part 2) Change of variables in triple integrals

Suppose we have the transform \(x = x(u, v, w), y = y(u, v, w)\) and \(z = z(u, v, w)\). Similarly, by the transformation, a small rectangular box is changed to a small parallelepiped. By computing the volume of the small parallelepiped using triple product, we find that the amplification factor from the volume \(du\, dv\, dw\) to the volume in \(xyz\) space is given by \(|J|\) where the Jacobian is given by

\[
J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix}
x_u & x_v & x_w \\
y_u & y_v & y_w \\
z_u & z_v & z_w
\end{vmatrix}
\]

The triple integral is then equation to

\[
\iiint_T f(x, y, z)\,dV = \iiint_D f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \,dudvdw
\]

\[
= \iiint_D f(x, y, z) \frac{1}{\left| \frac{\partial(u, v, w)}{\partial(x, y, z)} \right|} \,dudvdw
\]

Depending on which Jacobian is convenient, you can choose the suitable way to evaluate.

**Example:** Find the volume of the ellipsoid \( \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \leq 1 \).

**Solution.** We see if we do \(u = x/2, v = y/3, w = z/4\), then the region becomes a ball, which would be easy. The Jacobian is

\[
J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix}
2 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 4
\end{vmatrix} = 2 \times 3 \times 4.
\]

Hence, the volume is

\[
V = \iiint_T dV = \iiint_{u^2+v^2+w^2 \leq 1} |J|\,dudvdw
\]

\[
= \iiint_{u^2+v^2+w^2 \leq 1} 2 \times 3 \times 4\,dudvdw = 2 \times 3 \times 4 \times V(\text{unit ball}) = 2 \times 3 \times 4 \times \frac{4\pi}{3} \times 1^3.
\]

\(\Box\)
Example: Go over it if we have time. Find the volume of the solid torus obtained by revolving the disk \((x - b)^2 + z^2 \leq a^2\) in \(xz\) plane about \(z\)-axis.

Previously, we saw that the volume is \(V = 2\pi^2 a^2 b\) by Pappus's theorem. Now let's compute this using triple integral.

Solution. We first parametrize the torus. Suppose for a point \((x, y, z)\), the corresponding center is at \((b \cos u, b \sin u, 0)\) and the distance between them is \(w\). Assume the angle of the line segment from \(xy\) plane is \(v\). Then, 

\[ z = w \sin v \]

The projection of the point onto \(xy\) plane is \(b + w \cos v\) away from the origin. Hence, 

\[ x = (b + w \cos v) \cos u, \quad y = (b + w \cos v) \sin u, \quad z = w \sin v, \]

\[ 0 \leq w \leq a, \quad 0 \leq u, v < 2\pi. \]

The Jacobian is given by

\[
J = \frac{\partial (x, y, z)}{\partial (u, v, w)} = \begin{vmatrix}
(b + w \cos v)(-\sin u) & -w \sin v \cos u & \cos v \cos u \\
(b + w \cos v) \cos u & -w \sin v \sin u & \cos v \sin u \\
0 & w \cos v & \sin v
\end{vmatrix}
\]

\[ = (b + w \cos v)w \begin{vmatrix}
-\sin u & -\sin v \cos u & \cos v \cos u \\
\cos u & -\sin v \sin u & \cos v \sin u \\
0 & \cos v & \sin v
\end{vmatrix} = (b + w \cos v)w. \]

hence, the volume is

\[
V = \int_0^a \int_0^{2\pi} \int_0^{2\pi} |J| du dv dw = \int_0^a \int_0^{2\pi} \int_0^{2\pi} (b + w \cos v)wdudvdw = 2\pi^2 a^2 b.
\]

\[ \square \]

12.8. Cylindrical and spherical coordinates

For triple integrals, we usually use two kinds of special coordinates.

Cylindrical coordinates

Polar+\(z\) coordinates to represent any point in space: \((r, \theta, z)\). Draw a picture.

\[ x = r \cos \theta, y = r \sin \theta, z = z. \]

\[ r = \sqrt{x^2 + y^2}, \quad \tan(\theta) = y/x, \quad z = z \]

To cover the space once \(0 \leq \theta < 2\pi, 0 \leq r < \infty, -\infty < z < \infty. \)
Spherical coordinates

Draw a picture.
We use the distance to the origin $\rho$, the angle measured from $z$-axis $\phi$, and the polar angle $\theta$ (sometimes it’s called azimuthal angle in the spherical coordinate case). We have $(\rho, \phi, \theta)$.

Hence, we have

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.$$  

Further, we also have:

$$r = \sqrt{x^2 + y^2} = \rho \sin \phi.$$  

Comment: The names for the angles are different from the common ones in physics and engineering. In physics and engineering, many people use $\theta$ to mean the angle measured from $z$-axis and use $\varphi$ to mean the azimuthal angle: $(\rho, \theta, \varphi)$.

To cover the space once, $0 \leq \rho < \infty, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi$ (Some people answer $0 \leq \rho < \infty, 0 \leq \theta < \pi, 0 \leq \phi \leq 2\pi$. This is right but this convention is not used very often. We will not use this convention.)

Examples

Example: Write out the region $x^2 + y^2 + z^2 \leq a^2$ in both cylindrical and spherical coordinates.

Solution. In cylindrical: the boundary surface is $r^2 + z^2 = a^2$.

$$0 \leq r \leq a, 0 \leq \theta < 2\pi, -\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2};$$  

Alternatively, we have

$$-a \leq z \leq a, 0 \leq \theta < 2\pi, 0 \leq r \leq \sqrt{a^2 - z^2}$$

In spherical, the boundary surface is

$$\rho = a, \quad 0 \leq \rho \leq a, 0 \leq \phi \leq \pi, 0 \leq \theta < 2\pi.$$  

Example: Sketch the solid bounded by $z = r^2$ and $z = 8 - r^2$ where $r$ is the first cylindrical coordinate.
Solution. Both surfaces are surfaces of revolution about $z$ axis. They are radially symmetric.

When they intersect, $r^2 = 8 - r^2$ or $r = 2$.

$0 \leq r \leq 2, 0 \leq \theta < 2\pi, r^2 \leq z \leq 8 - r^2$.

Example: Describe the surface $\rho = 2 \cos \phi$.

Solution. This is $\sqrt{x^2 + y^2 + z^2} = 2z/\sqrt{x^2 + y^2 + z^2}$ or $x^2 + y^2 + z^2 = 2z$.

This is the sphere centered at $(0, 0, 1)$ with radius 1.