14.6 Triple integrals

Suppose \( T \) is a region in 3D space, contained in the domain of \( f(x, y, z) \). The triple integral of \( f \) over \( T \) is defined by the limit of Riemann sums:

\[
\int\int\int_T f(x, y, z)\,dV = \lim_{|P| \to 0} \sum_i f(x_i^*, y_i^*, z_i^*)\Delta V.
\]

Like the double integrals, we can change the triple integrals to iterated integrals. (Think about the Riemann sums. There is a 3D array of blocks. We can group the blocks in different ways. In principle, we have 6 ways, \( dx\,dy\,dz, dx\,dz\,dy, dy\,dx\,dz \) etc)

For a solid \( T \), the volume is

\[
V = \int\int\int_T dV.
\]

For a 3D region, given density \( \delta \) (mass per unit volume), we find similarly the total mass

\[
m = \int\int\int_T \delta dV.
\]

The centroid is \((\bar{x}, \bar{y}, \bar{z})\), where

\[
\bar{x} = \frac{1}{m} \int\int\int_T x\delta dV.
\]

The moments of inertia is

\[
I = \int\int\int_T p^2 dm = \int\int\int_T p^2 \delta dV.
\]

Note that the concrete formula for moment of inertia would be slightly different. For example, \( I_x \). The distance to \( x \)-axis is \( \sqrt{y^2 + z^2} \) instead of \(|y|\). Hence, we have

\[
I_x = \int\int\int_T (y^2 + z^2)\delta(x, y, z) dV.
\]

\( z \)-simple region means every vertical line intersects the region with a single line segment. Then, for given \( x, y \) the limits for \( z \) would depend on \( x, y \). We define \( x \)-simple or \( y \)-simple regions similarly.

**Example: Volume.** Set up the integral for the volume of the solid bounded by \( y + z = 4, y = 4 - x^2, y = 0, z = 0 \).
Solution. In double integral way, this is
\[ V = \iint_R (z_{\text{high}} - z_{\text{low}}) dA = \iint_R (4 - y - 0) dA. \]
The region \( R \) is given by \(-2 \leq x \leq 2, \ 0 \leq y \leq 4 - x^2\). Hence, the integral is
\[ V = \int_{-2}^{2} \int_{0}^{4-x^2} (4 - y - 0) dy dx. \]

In triple integral way,
\[ V = \iiint_T dV. \]
Let's figure out \( T \). The projection of the solid onto \( xy \) plane is the one determined by \( y = 4 - x^2 \) and \( y = 0 \). Hence, we have \(-2 \leq x \leq 2, \ 0 \leq y \leq 4 - x^2\). Then clearly, for given \((x, y)\), \(0 \leq z \leq 4 - y\). Hence, the volume is given by
\[ V = \int_{-2}^{2} \int_{0}^{4-x^2} \int_{0}^{4-y} dz dy dx = \int_{-2}^{2} \int_{0}^{4-x^2} (4 - y - 0) dy dx. \]

This means if we integrate \( z \) coordinate first, the triple integral can reduce to the double integral way exactly. However, the triple integral is more general as sometimes we integral \( x \) first or so (See Example 3 and 5 in your book).

Example: Set up the integrals for the mass of the pyramid \( T \) and the moment of inertial \( I_y \). \( T \) has vertices \((0, 0, 0)\), \((0, 3, 0)\), \((2, 0, 0)\) and \((0, 0, 6)\), and the density is given by \( \delta = z \).

For the plane, you can find the normal vector first using cross product. However, in this special case, we can construct it quickly: if a plane intersects with \( x, y, z \) at \((a, 0, 0)\), \((b, 0, 0)\), \((0, 0, c)\) respectively, then the equation is \( x/a + y/b + z/c = 1 \).

Solution. The equation for the top plane is \( x/2 + y/3 + z/6 = 1 \) or \( z = 6 - 3x - 2y \). The line in the \( xy \) plane is \( x/2 + y/3 = 1 \)(we simply set \( z = 0 \).
The region is \( 0 \leq x \leq 2, 0 \leq y \leq 3(1-x/2) = 3(2-x)/2, 0 \leq z \leq 6-3x-2y. \)
\[ m = \iiint_T z dV = \int_{0}^{2} \int_{0}^{3(2-x)/2} \int_{0}^{6-3x-2y} z \ dz \ dy \ dx. \]
The moment of inertia is
\[ I_y = \iiint_T (x^2 + z^2) \delta dV = \int_0^2 \int_0^{3(2-x)/2} \int_0^{6-3x-2y} (x^2 + z^2) zdV. \]

\[ \square \]

**Example:** Change the order. Write the following integral in the order \( dz \, dx \, dy \) and \( dx \, dy \, dz \):
\[ \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_{\sqrt{1-x^2-y^2}}^{\sqrt{1}-x} f(x, y, z) dz \, dy \, dx. \]

To solve it, we must have a clear picture what the solid is. According to the given bounds, \(-\sqrt{3}/2 \leq x \leq \sqrt{3}/2\) and \(-\sqrt{3}/4 \leq y \leq \sqrt{3}/4 - x^2\), we know that the projection of the solid into the \( xy \) plane is the disk given by \( x^2 + y^2 \leq 3/4 \). Having this in mind, let’s check the \( z \) direction. The lower bound is \( 1 - \sqrt{1-x^2-y^2} = z \) or \( (1-z)^2 + x^2 + y^2 = 1 \). Hence, the lower bound is the sphere centered at \((0, 0, 1)\) with radius 1. The upper bound is \( x^2 + y^2 + z^2 = 1 \) which is the unit sphere.

**Solution.** For \( dz \, dx \, dy \), we have
\[ \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_{\sqrt{1-x^2-y^2}}^{\sqrt{1}-x} f(x, y, z) dz \, dy \, dx. \]

For \( dx \, dy \, dz \), we must split the solid into two parts. One is for \( 0 \leq z \leq 1/2 \) and one is for \( 1/2 \leq z \leq 1 \). For \( 0 \leq z \leq 1/2 \), we need to look at the lower surface, ie, \( x^2 + y^2 = 1 - (1-z)^2 \). This determines \( dx \, dy \). For the upper half, we need to look at \( x^2 + y^2 = 1 - z^2 \). Hence, we have
\[ \int_{1/2}^{1} \int_{\sqrt{1-z^2}}^{\sqrt{1-(z-1)^2}} \int_{\sqrt{1-(z-1)^2-y^2}}^{\sqrt{1}-x} f(x, y, z) dx \, dy \, dz + \int_{1/2}^{1} \int_{\sqrt{1-z^2}}^{\sqrt{1-z^2-y^2}} \int_{\sqrt{1-z^2-y^2}}^{\sqrt{1-(z-1)^2-y^2}} f(x, y, z) dx \, dy \, dz \]

\[ \square \]

It is more challenging for change of order in 3D.