14.2 Double integrals for more general regions

If the region is not a rectangle, how do we compute the double integral? We can similarly change double integrals to iterated integrals as well. However, now, for a fixed $x$, the bounds of $y$ depend on $x$!

For vertically simple region $R$: for every vertical line, the intersection with the region is a single line segment (draw a picture).

$$a \leq x \leq b, \quad y_1(x) \leq y \leq y_2(x).$$

Similarly, we have horizontally simple region (draw a picture):

$$c \leq y \leq d, \quad x_1(y) \leq x \leq x_2(y).$$

**Exercise:** Recall that the region $x^2 + y^2 \leq 1$ can’t be written as $-1 \leq x \leq 1, -1 \leq y \leq 1$. Can you write it in a similar way like $\leq x \leq, \leq y \leq$?

$$-1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},$$

or

$$-1 \leq y \leq 1, -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}.$$

**Example:** Evaluate $\iint_R xy^2 \, dA$ in two ways where $R$ is the region bounded by $y = \sqrt{x}$ and $y = x^3$. $5/77$

**Solution.** The intersection of these two curves are $(0, 0)$ and $(1, 1)$.

By the picture if we express it in the vertically simple form, we have $0 \leq x \leq 1$ and $x^3 \leq y \leq \sqrt{x}$. This means we’ll integrate $y$ first:

$$\int_0^1 \int_{x^3}^{\sqrt{x}} xy^2 \, dy \, dx = \int_0^1 \frac{1}{3} xy^3 \bigg|_{y=x^3}^{\sqrt{x}} \, dx = \int_0^1 \left( \frac{1}{3} x^{5/2} - \frac{1}{3} x^{10} \right) \, dx$$

$$= \left[ \frac{1}{3} x^{7/2} - \frac{1}{3} \frac{1}{11} x^{11} \right]_0^1 = \frac{5}{77}.$$

If we express it in the horizontally simple form, we have $0 \leq y \leq 1, y^2 \leq x \leq y^{1/3}$.

$$\int_0^1 \int_{y^2}^{y^{1/3}} xy^2 \, dx \, dy = \int_0^1 \frac{1}{2} x^2 y^2 \bigg|_{x=y^2}^{x=y^{1/3}} \, dy = \int_0^1 \left( \frac{1}{2} y^{8/3} - \frac{1}{2} y^6 \right) \, dy$$

$$= \left( \frac{3}{4} \frac{1}{11} y^{11/3} - \frac{1}{4} \frac{1}{7} y^7 \right) \bigg|_0^1 = \frac{5}{77}.$$
Let’s look at more examples of double integrals on general regions to see more issues.

Example: Evaluate $\int_{R} (6x + 2y^2)\,dA$ over the region bounded by the parabola $x = y^2$ and $x + y = 2$.

Idea: If we do $y$ first, then, we’ll result in two pieces of regions. If we do $x$ first, then our life is easier!

Solution. We evaluate $x$ first. The intersections are $(1, 1)$ and $(4, -2)$. Then, $-2 \leq y \leq 1$, $y^2 \leq x \leq 2 - y$. The integral is
\[
\int_{-2}^{1} \int_{y^2}^{2-y} (6x + 2y^2)\,dxdy = \int_{-2}^{1} (3x^2 + 2xy^2)_{x=y^2} \,dy
= \int_{-2}^{1} (-5y^4 - 2y^3 + 7y^2 - 12y + 12) \,dy = 99/2.
\]

Example: Evaluate $\int_{0}^{\sqrt{\pi}} \int_{0}^{\sqrt{\pi}} \sin(x^2)\,dy\,dx$.

Idea: We see that we are unable to find the antiderivative of $\sin(x^2)$. Then, we may write the region in another way. Note that it’s very wrong to have $\int_{y}^{\sqrt{\pi}} \int_{0}^{\sqrt{\pi}} \sin(x^2)\,dy\,dx$. After evaluating on $y$, the $y$ variable should disappear.

Solution. The region given is $0 \leq y \leq \sqrt{\pi}$, $y \leq x \leq \sqrt{\pi}$. In another way, $0 \leq x \leq \sqrt{\pi}$, $0 \leq y \leq x$. Hence, the integral can be written as
\[
\int_{0}^{\sqrt{\pi}} \int_{0}^{x} \sin(x^2)\,dy\,dx = \int_{0}^{\sqrt{\pi}} x \sin(x^2)\,dx = \int_{0}^{\pi} \frac{1}{2} \sin(u)\,du = 1.
\]

Exercise: Evaluate $\int_{0}^{1} \int_{y^2}^{1} y(3x^2 + 1)^{1/3}\,dxdy$.

14.3 Areas, volumes and double integrals

Areas of a region in 2D plane

Recall that the area between two function graphs in the $xy$ plane is given by:
\[
A = \int_{a}^{b} (f_2(x) - f_1(x))\,dx.
\]
Using the double integral, it’s clear that

\[ A = \int\int_R dA. \]

The second formula is true by the Riemann sums: we sum up all the areas of the small regions and clearly it will be the total area. The first formula actually can be derived from the second if the region is vertically simple \( a \leq x \leq b, f_1(x) \leq y \leq f_2(x) \).

**Example:** Find the area of the region bounded by \( x = 2y^2 - 1 \) and \( x = y^4 \).

**Volumes in 3D space**

If \( f \) is nonnegative, the double integral is the volume under the surface and above the \( xy \) plane over the region \( R \). \( A(x) = \int_c^d f(x, y)dy \) is the area of the cross section. If we integrate the area of the cross section again, we get the volume we want.

If \( f = 1 \), the volume equals the base area times 1. Hence, \( \int\int_R 1dA \) is the area of \( R \). This is another explanation of the formula above.

How do we compute a volume between two surfaces \( z_1 = f_1(x, y) \) and \( z_2 = f_2(x, y) \)? Imagine the Riemann sum again, we find

\[ V = \int\int_R (z_2 - z_1)dA. \]

As you can imagine, the volume can also be evaluated by the triple integral

\[ V = \int\int\int_T dV, \]

we’ll come back later.

**Example:** Find the volume of the solid bounded by \( z = 6, z = 2y, y = x^2 \) and \( y = 2 - x^2 \).

**Solution.** Here, \( y = x^2 \) and \( y = 2 - x^2 \) are two cylinders with rulings parallel with \( z \). Hence, we only need to look at their traces in \( xy \) plane. The intersection of them \( x^2 = 1 \) or \( x = \pm 1, y = 1 \). For the two planes, clearly \( z = 2y \) is below \( z = 6 \) over this region. Hence, the volume is

\[ \int_{-1}^{1} \int_{x^2}^{2-x^2} (6 - 2y)dydx = \frac{32}{3}. \]
Example: Set up the integral for the volume of the solid bounded by $z = x^2 + y^2$ and $z = y$. (We’ll evaluate it later using polar coordinates.)

Solution. $z = x^2 + y^2$ is the paraboloid revolved from $z = x^2, y = 0$ about $z$-axis. We see that $z = x^2 + y^2$ is below $z = y$ for the region. hence, the volume is

$$V = \iint_R (y - (x^2 + y^2))dA.$$ 

Let’s determine the region. The intersection is $x^2 + y^2 = y, z = y$. The projection onto $xy$ plane is $x^2 + y^2 = y$. This is a circle centered at $(0, 1/2)$ with radius $1/2$. Hence, the region is the one bounded by this circle. We’ll evaluate this integral later. □