Math 212-Lecture 12

13.8(Part 2): The gradient vector

Recall the definition of gradient vector for \( f(x, y) \), where \((x, y)\) are the Cartesian coordinates, is given by

\[
\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle.
\]

We have similar expression for a function depends on \( n \) variables.

The chain rule for \( w = f(x(t), y(t), z(t)) \) can be written as

\[
\frac{dw}{dt} = \nabla f(r(t)) \cdot r'(t) = (\nabla f \cdot u)v = (D_uf)v = \frac{dw}{ds} \frac{ds}{dt}.
\]

The first term measures how fast the functions changes with respect to distance while the second term measures how fast the the distance changes as time increases. Hence, the time rate is equal to the directional derivative times speed.

**Example:** Suppose the temperature distribution of a room is given by

\[
f(x, y, z) = (9 - xy)e^{2-z}.
\]

A bug flies up at the origin and in the direction specified by \( v = \langle 1, 2, 2\sqrt{5} \rangle \) with speed \( v = 1.5 \). What is the initial rate of change of temperature will the bug feel? (Compare with the example in Lecture 10.)

**Solution.**

\[
\frac{df}{dt} = v \cdot D_uf.
\]

The speed is \( v = 1.5 \) while \( D_uf = \nabla f \cdot u \), which we computed in Lecture 10.

**Significance of the gradient vector**

Recall \( D_uf = \nabla f \cdot u = |\nabla f| \cos \theta \). When \( \theta = 0 \), the change of rate is the fastest.

1. **The gradient points the fastest increasing direction.**

When \( \theta = \pi/2 \), the rate of change is zero. In which direction is the changing rate zero? Along the level set! Hence, \( \nabla f \) is perpendicular with the level set. Actually, suppose \( r(t) \) is a curve in the level set \( f = k \). By the
chain rule \(0 = \frac{d}{dt} f(r) = \nabla f(r) \cdot r'(t)\). \(r'(t)\) is tangent to the level set and hence \(\nabla f\) is normal to the level set.

2. **The gradient vector is a normal vector of the level set.**

**Example:** Consider the level set \(F(x, y) = x^3 + y^3 - 3xy = 0\). At the two points \((\sqrt[3]{4}, \sqrt[3]{2})\) and \((0, 0)\), \(F_y = 0\), we can’t solve \(y\) in terms of \(x\). What happens at these two points? At the former, the tangent line is vertical. At the latter, there are two branches. By the geometric meaning, at \((0, 0)\), \(\nabla F = 0\).

**Tangent planes revisited**

Previously, we regarded the tangent plane as the linear approximation. Since any surface is the level set of some function. If we can find such a function, then, by taking the gradient of this function, we can find a normal vector of the plane.

**Example:** Find a normal vector of the graph of \(z = f(x, y)\).

**Solution.** \(F = f(x, y) - z\). Then, the graph is the zero level set of \(F\). The normal vector is simply \(\nabla F = \langle f_x, f_y, -1 \rangle\). From here, we can compute the tangent plane directly at \((a, b, f(a, b))\):

\[
\nabla F(a, b, f(a, b)) \cdot \langle x - a, y - b, z - f(a, b) \rangle = 0.
\]

Previously, we have the tangent plane to be \(z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)\). Hence, the normal vector at \((a, b, f(a, b))\) is \(n = \langle f_x(a, b), f_y(a, b), -1 \rangle\). This agrees. \(\square\)

**Example:** Consider \(F(x, y, z) = \ln(x^2 + y^2 + z^2)\). The level set \(F(x, y, z) = \ln(3)\) is a surface in 3D space passing through \((1, 1, 1)\). Find the tangent plane of this surface at \((1, 1, 1)\).

**Solution.** A normal vector is \(n = \nabla F(1, 1, 1)\). Hence, the tangent plane is

\[
\nabla F(1, 1, 1) \cdot \langle x - 1, y - 1, z - 1 \rangle = 0.
\]

Similar techniques can be applied to tangent lines of \(f(x, y) = k\).

### 14.1 Multiple integral: Double integral

For multiple integral, \(2D \to\) Double integral; \(3D \to\) triple integrals.
Let \( R \) be a region contained in the domain of the function \( f(x, y) \). The double integral is defined to be

\[
\iint_R f(x, y) \, dA = \lim_{|P| \to 0} \sum_i f(x_i^*, y_i^*) \Delta A_i,
\]

where \( P \) is a partition of the region \( R \) and \( A_i \) is the area of the \( i \)-th small region in the partition.

If \( f \geq 0 \), clearly \( f(x_i^*, y_i^*) \Delta A_i \) is the volume of the \( i \)-th cylinder. Hence, the double integral is the volume under the graph of \( z = f(x, y) \), above \( xy \) plane over region \( R \).

**Rectangular region**

Let \( R \) be a rectangle \([a, b] \times [c, d]\).

Draw a picture. Imaging we divide the rectangle into \( 3 \times 3 \) blocks. \( S = \sum_i \sum_j f(x_i, y_j) \Delta x \Delta y \). We can group each column first, or we can group each row first. Then, we have two ways to compute the final sum.

The double integral can be evaluated by the iterated integrals

\[
\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.
\]

**Example**: Let \( f(x, y) = 4x^3 + 6xy^2 \) and \( R = [1, 3] \times [-2, 1] \). Evaluate the double integral \( \iint_R f(x, d) \, dA \) using two different iterated integrals.

**Solution.** The first:

\[
\int_1^3 \int_{-2}^1 (4x^3 + 6xy^2) \, dy \, dx = \int_1^3 (4x^3 y + 2xy^3) \big|_{y=-2}^{y=1} \, dx = \int_1^3 (12x^3 + 18x) \, dx = (3x^4 + 9x^2) \big|_1^3 = 312.
\]

The second:

\[
\int_{-2}^1 \int_1^3 (4x^3 + 6xy^2) \, dx \, dy = \int_{-2}^1 (x^4 + 3x^2y^2) \big|_{x=1}^{x=3} \, dy = \int_{-2}^1 (80 + 24y^2) \, dy = (80y + 8y^3) \big|_{y=-2}^{y=1} = 312.
\]