Recall the linear approximation:
\[ \Delta f = f(x + \Delta x, y + \Delta y) - f(x, y) = f_x(x, y)\Delta x + f_y(x, y)\Delta y + \text{small}. \]
where \( f_x, f_y \) are continuous at \((x, y)\). When \( \Delta x \) and \( \Delta y \) are infinitesimal, then \( df = f_x dx + f_y dy \).

The differential is
\[ df = f_x dx + f_y dy = \langle dx, dy \rangle \cdot \langle f_x, f_y \rangle = dr \cdot \nabla f. \]
where \( \nabla f = \langle f_x, f_y \rangle \) is called the \textbf{gradient} vector. \( \langle dx, dy \rangle = dr \) is the differential of the position vector.

We have the product rule and chain rule:
\[ d(fg) = g df + f dg \]
\[ d(f(h)) = f'(h) dh. \]

\textbf{Example:} Find the differential for \( w = x \tan(yz) \).

Solution.
\[ dw = \tan(yz) dx + x d(\tan(yz)) = \tan(yz) dx + x \sec^2(yz) d(yz) \]
\[ = \tan(yz) dx + \sec^2(yz)(yz + zd) \]

We introduce differential simply to study how the infinitesimal change of the function value relies on the infinitesimal changes of the independent variables. However, \( df \) is actually a linear mapping of \( dx, dy \). In rigorous mathematical view, the differential is a mapping from the tangent space of the independent variables to the tangent space of the dependent variable, and \( dx, dy \) don’t even have to be small.

\textbf{Differentiability}

Let \( \mathbf{x} = \langle x, y \rangle \) or \( \mathbf{x} = \langle x, y, z \rangle \) and \( \mathbf{h} = \langle h_1, h_2 \rangle \) or \( \mathbf{h} = \langle h_1, h_2, h_2 \rangle \). \( f \) is said to be differentiable at \( \mathbf{x} \), if there exists a vector \( \mathbf{c} \) such that
\[ \lim_{|\mathbf{h}| \to 0} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - \mathbf{c} \cdot \mathbf{h}}{|\mathbf{h}|} = 0. \]
Differentiability means that we can approximate $f$ with a linear function. Hence, for the linear approximation to exist, we do not need to require $f_x, f_y$ to be continuous. We only need $f$ to be differentiable there.

Facts: 1. If $f_x, f_y, f_z$ are continuous (In this case, $f$ is called continuously differentiable), $c = \nabla f$ evaluated at $x$ works by the linear approximation. Hence, the function is differentiable.

2. Differentiability implies that partial derivatives exist and the function is continuous at that point $x$. Further, $c = \nabla f$. (Recall that the existence of partial derivatives doesn’t imply the continuity.) However, the partial derivatives may not be continuous at the point.

We see that the existence of partial derivatives doesn’t imply the differentiability (It doesn’t even imply the continuity of the function), but continuity of the partial derivatives is something stronger than the differentiability.

**Tangent planes and tangent lines as graphs of linear approximations**

Suppose that $f$ is differentiable at $(a, b)$, then the graph of the linear approximation is a plane, which is tangent to the function graph. To see this, we check that every tangent line of the surface surface $z = f(x, y)$ at $(a, b, f(a, b))$ is in the plane.

By linear approximation, the plane is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Consider the $x$ curve, $z = f(x, b)$. The tangent line at $(a, b, f(a, b))$ is given by $z = f_x(a, b)(x - a)$. This line is in the plane. You can check other tangent lines as well.

**Example:** Find the tangent plane of $z = e^{-x^2 - y^2}$ at $(0, 0, 1)$

**Solution.** Let $f(x, y) = e^{-x^2 - y^2}$.

$$f(0, 0) = 1,$$

$$f_x(x, y) = e^{-x^2 - y^2}(-2x) \Rightarrow f_x(0, 0) = 0$$

$$f_y(x, y) = e^{-x^2 - y^2}(-2y) \Rightarrow f_y(0, 0) = 0.$$

Hence

$$z - 1 = 0(x - 0) + 0(y - 0), \rightarrow z = 1.$$
Tangent planes of level sets

We know the level set \( f(x, y) - f(a, b) = 0 \) is a curve in \( xy \) plane. Then, the tangent line of the curve at \((a, b)\) is also given by the graph of the linear approximation:

\[
f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) - f(a, b) = 0.
\]

or

\[
f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0.
\]

Example: Consider \( F(x, y, z) = \ln(x^2 + y^2 + z^2) \). The level set \( F(x, y, z) = \ln(3) \) is a surface in 3D space passing through \((1, 1, 1)\). Find the tangent plane of this surface at \((1, 1, 1)\).

The idea is to use linear approximation. \( F(x, y, z) - F(1, 1, 1) \approx F(1, 1, 1) + F_x(1, 1, 1)(x - 1) + F_y(1, 1, 1)(y - 1) + F_z(1, 1, 1)(z - 1) - F(1, 1, 1) = 0 \), or \( F_x(1, 1, 1)(x - 1) + F_y(1, 1, 1)(y - 1) + F_z(1, 1, 1)(z - 1) = 0 \)

Solution.

\[
F_x(x, y, z) = \frac{2x}{x^2 + y^2 + z^2} \Rightarrow F_x(1, 1, 1) = \frac{2}{3}
\]
\[
F_y(x, y, z) = \frac{2y}{x^2 + y^2 + z^2} \Rightarrow F_y(1, 1, 1) = \frac{2}{3}
\]
\[
F_z(x, y, z) = \frac{2z}{x^2 + y^2 + z^2} \Rightarrow F_y(1, 1, 1) = \frac{2}{3}
\]

The tangent plane is hence

\[
\frac{2}{3}(x - 1) + \frac{2}{3}(y - 1) + \frac{2}{3}(z - 1) = 0.
\]

\[\square\]

13.8(Part 1): Directional Derivatives

We now assume that \( f \) is differentiable at a point \( x = (x, y) \), which means \( \nabla f(x, y) = \nabla f(x) \) exists and

\[
\lim_{|h| \to 0} \frac{f(x + h) - f(x) - \nabla f(x) \cdot h}{|h|} = 0.
\]
As we have seen, $f_x$ and $f_y$ are the changing rates in the direction of $x$ and $y$. How about an arbitrary direction? Let $u$ be a unit vector. The directional derivative in the direction of $u$ is defined to be:

$$D_uf(x) = \lim_{h \to 0} \frac{f(x + hu) - f(x)}{h}.$$ 

$h$ is the distance from $x$ in the direction $u$. The directional derivative therefore is the instantaneous rate of change with respect to the distance in the direction of $u$.

How do we compute?

In the expression for differentiability, if we substitute, $h = hu$, we then see clearly that

$$D_uf(x) = \lim_{h \to 0} \frac{f(x + hu) - f(x)}{h} = \frac{\nabla f(x) \cdot hu}{h} = \nabla f(x) \cdot u = |\nabla f| \cos \theta.$$

Clearly, in the direction $u = \nabla f/|\nabla f|$ when $\theta = 0$, the function values increase the fastest and in the direction $u = -\nabla f/|\nabla f|$ when $\theta = \pi$, the function values decrease the fastest.

**Example:** Suppose the temperature distribution of a room is given by

$$f(x, y, z) = (9 - xy)e^{2-z}.$$ 

A bug flies up at the origin and in the direction specified by $v = \langle 1, 2, 2\sqrt{5} \rangle$. What is the initial rate of change of temperature with respect to distance will the bug observe?

**Solution.** By the meaning of directional derivative, the rate of change with respect to distance is just $D_uf$ where $u$ is the unit vector given by

$$u = \frac{1}{|\langle 1, 2, 2\sqrt{5} \rangle|} \langle 1, 2, 2\sqrt{5} \rangle = \frac{1}{5} \langle 1, 2, 2\sqrt{5} \rangle.$$ 

We see $f_x = -ye^{2-z}$, $f_y = -xe^{2-z}$ and $f_z = -(9 - xy)e^{2-z}$. Hence,

$$\nabla f(0, 0, 0) = \langle 0, 0, -9e^2 \rangle.$$ 

The rate of change is $\nabla f(0, 0, 0) \cdot u = -\frac{18\sqrt{5}}{5}e^2$. □