Quiz 7

Consider the field \( \mathbf{F} = (y^4 - z^8 + x^2, -2xy + \cos(x^3) - z^8, y^3) \). Suppose \( S \) is the part of the surface \( x^2 + y^2 + (z - 2)^2 = 5 \) below \( xy \) plane with the normal being upward. Compute the flux of \( \mathbf{F} \) through \( S \) without really doing actual integration. (Hint: Symmetry for the double integral you obtain.)

The surface is not closed and we can’t apply the Divergence theorem.

Note

\[
\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \left( y^4 - z^8 + x^2 \right)_x + \left( -2xy + \cos(x^3) - z^8 \right)_y + (y^3)_z = 2x - 2x = 0.
\]

The vector field then satisfies the surface independence property.

\[
\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} dS.
\]

We use the surface in \( xy \) plane with the same boundary which is the disk inside \( x^2 + y^2 + 4 = 5, z = 0 \) or the unit disk in \( xy \) plane. Hence,

\[
\iint_D \mathbf{F} \cdot \mathbf{k} dA = \iint_D y^3 dA = 0
\]

where \( D \) is the unit disk. The final answer is zero by symmetry because \( y^3 \) is an odd function in \( y \) and the region is symmetric about \( x \)-axis.
Consider the vector field \( \mathbf{F} = (1, x, y^3 + e^{x^8 + 101x^3}) \). \( S \) is the part of the surface \( (x - z)^2 + y^2 + (z - 1)^2 = 2 \) above \( xy \) plane. The normal is upward. Compute the surface integral

\[
\iint_{S} \left( \nabla \times \mathbf{F} \right) \cdot \mathbf{n} \, dS
\]

**Hint:** For this surface, the cross section in \( z = z_0 \) is a circle for \( 0 < z_0 < 1 + \sqrt{2} \). In the plane \( z = 1 + \sqrt{2} \), \( S \) is a point. Hence, \( S \) only has a boundary in \( xy \) plane. After you get the line integral, you can switch it to a double integral by Green’s.

The surface has a boundary \( C : x^2 + y^2 + 1 = 2 \) in \( z = 0 \). The boundary should be oriented counterclockwise if viewed from above. Then, we have by **Stokes theorem**

\[
\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \oint_{C} 1 \, dx + x \, dy + (\ldots)dz = \oint_{C} 1 \, dx + x \, dy
\]

For this, you can use the parametrization or Green’s. By Green’s, it is

\[
\iint_{D} dA = \text{Area}(D) = \pi.
\]

Some people used the surface independence for \( \text{curl}(\mathbf{F}) \). That is fine as well, but **say what you used**.