Quiz 3

1. (3 pts) Answer True or False (No need to explain):

- \[ \int_0^1 \int_{x^3}^x (3x + y) dy dx = \int_{x^3}^1 (3x + y) dx dy. \]
- \[ \int_a^b f(x) f(y) dy dx = (\int_a^b f(x) dx)^2. \]

For the first, it’s false. \( \int_b^a \int_d^c = \int_c^d \int_a^b \) is true only if \( a, b, c, d \) are constants where the region is a rectangle. When the region is a general region, we must rearrange. In this case, it would be \( \int_0^1 \int_{\sqrt[3]{y}}^1 (3x + y) dy dx \)

For the second, it’s true. When we integrate \( x \), \( f(y) \) is a constant. Hence, we can pull out \( f(y) \) and have

\[ \int_a^b f(y) \int_a^b f(x) dx dy. \]

However, \( \int_a^b f(x) dx \) is a number independent of \( y \), we can further pull it out and have

\[ (\int_a^b f(x) dx) \int_a^b f(y) dy. \]

Both of them are equal to \( F(b) - F(a) \) if \( F \) is one member of the antiderivative. Hence, the right hand side is correct.

2. Consider the lamina \( R = \{(x, y) : 0 \leq y \leq 1, y^2 \leq x \leq 1\} \). Suppose the density of mass of this lamina is given by \( \delta(x, y) = y \sin(x^2) \). Find the total mass of this lamina.

The total mass is

\[ m = \int_0^1 \int_{y^2}^1 y \sin(x^2) dx dy. \]

It’s hard to evaluate directly. We change the order of integration:

\[ \int_0^1 \int_0^{\sqrt{x}} y \sin(x^2) dy dx = \int_0^1 \frac{1}{2} x \sin(x^2) dx. \]

At this point, we do \( u \)-substitution \( u = x^2 \) and the integral is reduced to

\[ \int_0^1 \frac{1}{4} \sin(u) du = \frac{1}{4}(1 - \cos(1)). \]