Practice problems

1. Consider a right circular cone of uniform density. The height is $H$. Let’s say the distance of the centroid to the base is $d$. What is the value $d/H$?

2. Set up the integral without evaluation. The volume inside $(x - 1)^2 + y^2 + z^2 = 1$, below $z = \sqrt{3}r$ but above $z = -r$.

3. Set up the integral for the moment of inertia about $z$ axis inside both $\rho = 2$ and $r = 2 \cos \theta$, outside $r = 1$ and above $xy$ plane. The density is $\delta = \sqrt{x^2 + y^2 + z^2}$.

4. Find the centroid of the ice-cream cone enclosed by $x^2 + y^2 + z^2 = 1, z \geq 0$ and $z = r - 1, z \leq 0$. Suppose the density is $\delta = 1$ (Hint: break the region into two parts. One for cylindrical and one for spherical.)

5. Find the total mass of the ice-cream cone inside $x^2 + y^2 + (z - 1)^2 = 1$ and above $z = \sqrt{3}r$, assuming the density is $\delta = 1$.

6. Set up the integral for the volume outside $r = 1$ inside $\rho = 2$ in both cylindrical and spherical coordinates.

7. (The practice problems for 14.8 will be together with the one for surface integrals.)

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1. Line integrals

(a) Parametrization

- Parametrize the curve $y = x^2$
- Parametrize $x^2 + 4y^2 = 1$
- Parametrize the boundary of the region bounded by $x$-axis, $y = \sqrt{x}$ and $x = 1$.
- Parametrize the ellipse formed by the intersection of $x^2 + y^2 = 1$ and $x + z = 0$.

(b) Usual line integrals (2 types)

- Consider the curve $x^2/4 + y^2 = 1$ with $x \geq 0, 0 \leq y \leq 1/2$. If the density (per unit length) is $\delta = y/x$, compute the moment of inertia $I_y$. 

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• Compute the line integral of \( \mathbf{F} = \langle 3y, -2x \rangle \) over the curve \( y = x^2 \) for \( 0 \leq y \leq 1 \) oriented from right to left.

• Let \( \mathbf{r} = (t^3, t^2, t), 0 \leq t \leq 1 \). Compute \( \int_C \mathbf{F} \cdot \mathbf{T} \, ds \) where \( \mathbf{F} = \langle e^{yz}, 0, ye^{yz} \rangle \)

(c) Conservative field.

• Compute the line integral \( \int_C (x^3 + y) \, dx + xdy \) where \( C \) is the curve jointing \((0,0)\) and \((1,1)\). Justify your answer.

• Let \( C \) be \( \mathbf{r}(t) = (\ln(1+t^3), t^3 + 1, t^{100}), 0 \leq t \leq 1 \). Compute \( \int_C xdy + ydx + dz \)

• Show that \( \mathbf{F} = (3y^3 - 10xz^2)i + 9xy^2j - 10x^2zk \) is irrotational and thus conservative. Find a potential function \( \phi \).

• \( \int_C \mathbf{F} \cdot \mathbf{T} \, ds \) where \( \mathbf{F} = \langle xe^{xz} + e^x, 2yz, xe^{xz} + y^2 \rangle \). \( \mathbf{r} = \langle e^t, e^t, t^4 \rangle, t \in [0,1] \)

• Let \( C \) be \( \mathbf{r}(t) = (\cos^4 t, \sin^4 t, 7), 0 \leq t < 2\pi \) and \( \mathbf{F} = \langle x^3 - z, y^3, y + z^3 \rangle \). Compute the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \). (Hint: Split out a conservative field. The integral of the one you split will be zero since it is on a closed curve.)

2. Green’s theorem

(a) Circulation and flux

• Compute the line integral of \( \langle x^2y + xe^{x^3}, xy - \sin^2(e^y) \rangle \) over the rectangle with vertices \((0,0), (2,0), (0,3)\) and \((2,3)\) oriented counterclockwise.

• Compute the line integral \( \oint_C P \, dx + Q \, dy \) where \( P = xy, Q = x^2 \) and \( C \) is the loop of the curve in the first quadrant whose polar equation is \( r = \sin(2\theta) \).

• Let \( \mathbf{F} = xy^2 \mathbf{i} + x^2y \mathbf{j} \). Let \( C \) be the union of \( x\)-axis(|\( x | \leq 2) \) and \( y = \sqrt{4 - x^2} \), oriented counterclockwise. Compute the flux \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds \) in two ways.

• If \( \mathbf{v} = (y^2, xy) \), and \( C \) is the ellipse \( x^2/9 + y^2/4 = 1 \), compute the circulation \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) in two ways.

(b) Suppose \( \mathbf{v} = \langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \rangle \). Suppose \( C \) is a simple closed curve in the plane, counterclockwise. What values can \( \oint_C \mathbf{v} \cdot \mathbf{n} \, ds \) be? (Hint: This field is divergence free.)

• Suppose \( \mathbf{F} = \langle \frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle \). Suppose \( C \) is a simple closed curve in the plane, counterclockwise. What values can \( \oint_C \mathbf{F} \cdot \mathbf{T} \, ds \) be? (Hint: This field is curl free or \( P_y = Q_x \).)
(c) 
- Compute the area bounded by \( y = x^3 \), \( y \)-axis and \( y = 1 \) using both line integral and double integral.
- Find the area of the region enclosed by \( r(t) = (\sin(2t), \sin(t)) \) above \( x \)-axis.
- Find the area of the region enclosed by one arch of the cycloid \( x(t) = a(t - \sin(t)), y(t) = a(1 - \cos(t)) \) and \( x \)-axis.

3. Surface integrals (2 types)

(a) Parametrization, surface area element and the surface integral of a function
- Parametrize the surface \( y = f(x, z) \). Use this to compute the area of the plane \( y = 2x + 2z + 1 \) inside \( x^2 + z^2 = 1 \).
- Consider the surface of revolution obtained by revolving \( x = f(z) \) about \( z \) axis. Parametrize this surface.
- Consider the fence \( S: x = 2\sin(t), y = 8\cos(3t), 0 \leq t < 2\pi \) and \( 0 \leq z \leq 2 \). Set up the surface integral \( \iint_S 2yzdS \).

(b) Flux
- Compute the flux of \( G = (2x, x-y, y+z) \) through the surface \( S \), which is the portion of the plane \( 2x - 3y + 5z = 0 \) inside \( x^2 + y^2 = 1 \) oriented upward. Can we use Stokes's theorem here? Why?
- Parametrize the upper hemi-sphere with radius 1. Then compute the flux of \( F = (x^2, 0, 0) \) across it.
- \( \iint_S F \cdot ndS \) where \( F = (y, -x, z) \) and \( S \) is the surface \( z = \theta, 0 \leq \theta \leq \pi \) and \( 1 \leq x^2 + y^2 \leq 4 \).
- Compute the flux of \( F = (2, 2, 3) \) across the surface \( S: r(u, v) = (u + v, u - v, uv), 0 \leq u, v \leq 1 \)

4. Stokes Theorem

- Let \( F = z \hat{i} + x \hat{j} - y \hat{k} \). Let \( C \) be the intersection between \( x^2 + y^2 = 1 \) and \( z + y = 3 \) oriented counterclockwise if viewed from above. Compute \( \int_C F \cdot Tds \) in two ways.
  If the vector field is \( F = (z + x^3 + \sin(x), x + e^{y^2} + e^y, -y - \cos(8z) - z^{100}) \), can you reduce this to the problem here? Why?
- Let \( F = (3y, -2x, x^2y^2z^2) \). Let \( S \) be the surface \( x^2 + y^2 + (z - 1)^2 = 2 \) above \( xy \) plane, oriented upward. Compute
  \[ \iint_S (\nabla \times F) \cdot n dS \]
• Suppose \( F = (xy^2z^2 + y, x^2yz^2 + z, x^2y^2z + z) \). \( C \) is the hexagon with vertices \((2,0,1), (1,0,2), (0,1,2), (0,2,1), (1,2,0)\) and \((2,1,0)\), which are all in the plane \( x+y+z = 3 \). Compute the circulation of the field over this curve. (Hint: The best way is to split a conservative field first. Anyway, using Stokes theorem directly will give you the same answer.)

• Compute the line integral of \( \vec{G} = (x^2 - 2y, 2e^y - z, z^3 + 3x) \) along the curve \( r(t) = \langle \cos t, \sin t, \cos t \sin t \rangle \) where \( t : 0 \to 2\pi \).

5. Divergence theorem

(a) Computation and applications

• Compute the flux \( \iint_S F \cdot \mathbf{n}dS \) where \( F = (y^3 + z^2, xy - xz^2, xe^y) \). \( S \) is the boundary of the solid \( x+y+z \leq 1, x, y, z \geq 0 \), with \( \mathbf{n} \) being the outer normal.

• Let \( F = (x + e^{8xz}, y + 3y^2 + \ln(x^8 + 1000), z + \cos(xy)) \). Let \( T \) be the upper hemi-ball with radius 1. Compute the flux of this field out of \( T \).

• Suppose \( \mathbf{v} \) is the gravitational field generated by a cloud of mass. The physical law tells us that \( \iiint_S \mathbf{v} \cdot \mathbf{n}dS = -4\pi Gm \) where \( m \) is the total mass inside \( S \). Suppose the density of the mass is \( \delta \). Then, the total mass is \( m = \iiint_T \delta dV \). Using the divergence theorem, show that \( \delta = -\frac{1}{4\pi G} \nabla \cdot \mathbf{v} \). If the gravitational field is given by \( \mathbf{v} = \langle 3e^x + 4y^2, 2y^2 + ey^2, z^2 + xyz \rangle \). Compute \( \delta(0,0,1)/\delta(0,0,0) \)

(b) Surface independence

• Let \( F = -\frac{GM}{r^3} \mathbf{r} \) be the gravitational field generated by the Earth. \( S \) is any surface that does not go across the Earth. Find the flux \( \iint_S F \cdot \mathbf{n}dS \)

• Consider the surface \( z = (x^2 + y^2 - 1)(x^4 + y^4 + 1) \) for \( z \leq 0 \), oriented upward. Compute \( \iint_S F \cdot \mathbf{n}dS \) where \( F = (xy^3 + y^2z, x^2z^2 - x^2y, z^2 - zy^3) \).

• Let \( S \) be \( r(t,z) = \langle (1 - z)^3 \cos t, (1 - z)^3 \sin t, z \rangle \) and \( 0 \leq t < 2\pi, 0 \leq z \leq 1 \). The normal is upward. Compute the flux of \( \vec{F} = (y^2z - z^2, 4 - xy, 3 + xz) \) through this surface.