13.4: Partial Derivatives

Definition

For a single-variable function \( z = f(x) \), the derivative is \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).

For a function \( z = f(x,y) \) of two variables, to define the derivatives, we hold one variable fixed:

\[
fx(x,y) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h},
\]

\[
fy(x,y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.
\]

They are called the ‘partial derivatives’. We can similarly define partial derivatives for a function of three variables \( w = f(x,y,z) \).

Example 1: consider

\[
f(x,y) = \begin{cases} 
    xy & (x,y) \neq (0,0) \\
    x^2 + y^2 & x = y = 0
\end{cases}
\]

The function is not continuous at \((0,0)\) but we can compute the partial derivatives at \((0,0)\) (this means that the function is continuous along \(x\) and \(y\) directions but not continuous along an arbitrary path.). Find \(fx(0,0)\) and \(fy(0,0)\).

Solution.

\[
fx(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.
\]

Similarly, \(fy(0,0) = 0\).

Comment: From this example, we see that the existence of partial derivatives doesn’t imply the continuity of the function itself. It only implies that the function is continuous along \(x\) or \(y\) direction. Later, we’ll see that differentiability instead of the existence of partial derivatives implies continuity.

When we compute the partial derivatives, the other variable is regarded as a parameter (constant).

Example 2: Consider \( w = f(x,y,z) = xyz \ln(xyz) \). Compute \(fx\) and \(fz\).
Solution.

\[ f_x = (xyz)_x \ln(xyz) + (xyz)(\ln(xyz))_x = yz \ln(xyz) + yz \cdot \frac{1}{xyz} \cdot yz = yz(\ln(xyz) + 1). \]

Similarly, you can compute \( f_y \) and \( f_z \).

\[ \square \]

Interpretation

(Draw the picture.)

- Instantaneous rates of change with respect to \( x \) or \( y \)
- Slope of the line tangent to the \( x \)-curve or \( y \)-curve

Planes tangent to the surface

Suppose that \( f_x \) and \( f_y \) of the function \( f(x, y) \) are continuous at \((a, b)\), then there is a plane tangent to the surface \( z = f(x, y) \) at \((a, b, f(a, b))\), which means that every tangent line of the surface at \((a, b, f(a, b))\) is in the plane.

Suppose the plane is \( A(x - a) + B(x - b) + C(z - z_0) = 0 \) where \( \langle A, B, C \rangle \) is a normal vector. For the tangent plane of a function, \( C \neq 0 \) (why?), then

\[ z - z_0 = p(x - a) + q(y - b). \]

Since the plane contains \((a, b, f(a, b))\), \( z_0 = f(a, b) \). Since the tangent line of the \( x \)-curve is in the plane, we must have \( p = f_x(a, b) \). Similarly, \( q = f_y(a, b) \).

**Example 3:** Find the tangent plane of \( z = e^{-x^2 - y^2} \) at \((0, 0, 1)\)

Solution. Let \( f(x, y) = e^{-x^2 - y^2} \).

\[ f(0, 0) = 1, \]
\[ f_x(x, y) = e^{-x^2 - y^2}(-2x) \Rightarrow f_x(0, 0) = 0 \]
\[ f_y(x, y) = e^{-x^2 - y^2}(-2y) \Rightarrow f_y(0, 0) = 0. \]

Hence

\[ z - 1 = 0(x - 0) + 0(y - 0), \Rightarrow z = 1. \]

\[ \square \]
High order derivatives

\[ f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial^2 f}{\partial x^2} \]
\[ f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial^2 f}{\partial y \partial x} \]
\[ f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial^2 f}{\partial x \partial y} \]

Clairaut’s theorem:

**Theorem 1.** If both \( f_{xy} \) and \( f_{yx} \) are defined in a ball containing \((a, b)\) and they are continuous at \((a, b)\), then
\[ f_{xy}(a, b) = f_{yx}(a, b). \]

If they are not continuous, it’s possible that they are not equal (see Problem 74).

**Exercise:** Prove Clairaut’s theorem.

**Proof.**

\[ f_{xy}(a, b) = \lim_{k \to 0} \frac{1}{k} (f_x(a, b + k) - f_x(a, b)) \]
\[ = \lim_{k \to 0} \lim_{h \to 0} \frac{1}{h} \left([f(a + h, b + k) - f(a, b + k)] - [f(a + h, b) - f(a, b)]\right) \]
\[ = \lim_{k \to 0} \lim_{h \to 0} \frac{1}{hk} \left([f(a + h, b + k) - f(a + h, b)] - (f(a, b + k) - f(a, b))\right) \]

Now, using that \( f_y \) is defined in the neighborhood, applying the mean value theorem to \( g(y) = f(a + h, y) - f(a, y) \), we have
\[ \lim_{k \to 0} \lim_{h \to 0} \frac{1}{h} \left[f_y(a + h, b + \xi(k)) - f_y(a, b + \xi(k))\right] \]

Applying the mean value theorem again,
\[ \lim_{k \to 0} \lim_{h \to 0} f_{yx}(a + \eta(h), b + \xi(k)) \]
using the fact that \( f_{yx} \) is continuous, we are done. \( \square \)

**Example:** Given \( P = 2x + \cos x \sin y \) and \( Q = \sin x \cos y + e^y \), can you find \( f \) such that \( f_x = P \) and \( f_y = Q \)? If it’s impossible, explain why; if it is possible, find such one \( f \).
Solution. Suppose such one \( f \) exists. Then,

\[
\begin{align*}
    f_{xy} &= P_y = \cos x \cos y, \\
    f_{yx} &= Q_x = \cos x \cos y.
\end{align*}
\]

Both of them are continuous. Hence, if the \( f \) exists, we must have \( P_y = Q_x \).

This is true for this example. Hence, \( f \) should exist (there is another thing about the domain, but we'll come back to this very later).

Then, to recover \( f \),

\[
f = \int P\,dx = x^2 + \sin x \sin y + C(y).
\]

Then, \( f_y = Q \) tells us that

\[
\sin x \cos y + C'(y) = \sin x \cos y + e^y.
\]

hence, \( C'(y) = e^y \) or \( C(y) = e^y + C \). Hence,

\[
f(x, y) = x^2 + \sin x \sin y + e^y + C.
\]

Another way is to use the differentials to find \( f \). □

Example: Suppose the high order partial derivatives of \( f(x, y, z) \) are all continuous. Show that

\[
f_{xxz} = f_{zxx}.
\]

Comment: If the derivatives are continuous, the order of taking the derivatives is not important.

Proof. First of all, let \( g = f_x \). Since the derivatives are continuous, by the theorem, \( g_{xz} = g_{zx} \) or \( f_{xxz} = f_{zxx} \). Now, as functions, \( f_{xz} = f_{zx} \) by the theorem again. Hence, \( f_{xxx} = f_{zxx} \). □