Math 212-Lecture 4

12-4: Lines and planes

Lines
Given a point on the line $P_0(x_0, y_0, z_0)$ and a vector $v = (a, b, c)$ that is parallel with the line, we determine that any point on the line $P(x, y, z)$ satisfies the relation $\overrightarrow{P_0P} = tv$. The position vector of $P$ is given by

$$r = \overrightarrow{OP_0} + tv.$$  

This is called the vector equation of the line.

Parametric equations

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$ 

Symmetric equations (assume $a, b, c$ are all nonzero):

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$ 

Exercise: Suppose we have a line given by $r = r_0 + tv$. Find the distance from $Q(x, y, z)$ to the line.

Example: (1). Find the parametric and symmetric equations of the line $L$ through $P_0(3, 1, -2)$ and $P_1(4, -1, 1)$. (2). Find a parametric equation for the line segment $P_0P_1$.

(For (2), find the range of $t$)

Example: Consider the skew lines $L_1 : x = 1 + t, y = -1 - t, z = 2 - 3t$ and $L_2 : x = t, y = t + 1, z = 2 - t$. What is the distance between them?

Planes

Suppose $n = (a, b, c)$ is perpendicular with Plane $P$ (this is called normal vector) and $P_0(x_0, y_0, z_0)$ is on the plane. Then, $\overrightarrow{P_0P} \perp n$. The vector equation:

$$(r - r_0) \cdot n = 0,$$

where $r = \overrightarrow{OP}$ and $r_0 = \overrightarrow{OP_0}$.

Scalar equation:

$$\displaystyle a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$
Angle between two planes

If two planes have normal vectors \( m \) and \( n \) respectively, the angle between them satisfies
\[
\cos \theta = \frac{|\mathbf{m} \cdot \mathbf{n}|}{|\mathbf{m}| |\mathbf{n}|}.
\]

Exercise: Draw a picture to illustrate that this formula is true.

**Example:** Suppose \( A(1, 0, -1), B(3, 3, 2), C(4, 5, -1) \) and \( D(0, 0, 1) \).

- Find the scalar equation for the plane \( ABC \)
- Find the distance from point \( D \) to the plane \( ABC \)

**Solution.** (1). A normal vector is \( \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (-15, 9, 1) \). If we use the point \( A(1, 0, -1) \), then the equation will be \(-15(x - 1) + 9(y - 0) + 1(z + 1) = 0 \) or \(-15x + 9y + z + 16 = 0 \).

(2). \( d = |\overrightarrow{AD} \cdot \mathbf{n}|/|\mathbf{n}|. \)

**Example:** The two lines \( \frac{x - 1}{2} = \frac{y - 3}{3} = z - 1 \) and \( \frac{x - 1}{2} = \frac{y - 3}{2} = z - 1 \) intersect at a point. Find the scalar equation of the plane that contains these two lines.

**Solution.** Let \( \mathbf{v}_1 = (2, 3, 1) \) and \( \mathbf{v}_2 = (-1, 2, 1) \). A normal vector is \( \mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2. \)

12-5: Curves and motion in space

A point moves in space and it traces out a curve. The coordinates of that point can be written as
\[
x = x(t), \quad y = y(t), \quad z = z(t).
\]

In other words, the position vector can be written as
\[
\mathbf{r}(t) = (x(t), y(t), z(t)) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.
\]

Hence, a curve is described by a vector-valued function.
Continuity, Differentiation, Integration

We can simply look at each component.
For example, the derivative is defined to be
$$r'(t) = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}.$$ 
We now show that $r'(t) = (x'(t), y'(t), z'(t))$.

Proof.

$$r'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ (x(t + \Delta t)i + y(t + \Delta t)j + z(t + \Delta t)k) - (x(t)i + y(t)j + z(t)k) \right]$$

$$= \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}i + \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t}j + \lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t}k$$

$$= x'(t)i + y'(t)j + z'(t)k.$$

Another example: the integration $\int_a^b r(t)dt$ is defined to be

$$\int_a^b r(t)dt = \lim_{\Delta t \to 0} \sum_{i=1}^{n} r(t_i^*) \Delta t,$$

where we divide $[a, b]$ into $n$ subintervals and $t_i^*$ is a sample point from the $i$-th subinterval.

Exercise: Read P809 and understand why we can integrate each component to get the integral. If $r = u(t)a(t) + v(t)b(t)$, is it true or false that $\int_0^1 r(t)dt = \int_0^1 u(t)dt + \int_0^1 v(t)dt$?

(The answer is false. Here, the two unit vectors are changing.)

Example: Given $r(t) = ie^t - jte^{-t^2}$, evaluate $r'(0)$ and $\int_0^1 r(t)dt$.

Solution. $r'(t) = e^t i + ((-e^{-t^2} + 2te^{-t^2})j$ and hence $r'(0) = i - j$.

$$\int_0^1 r(t)dt = i \int_0^1 e^t dt + j \int_0^1 (-te^{-t^2}) dt = (e - 1)i + \frac{1}{2}(e^{-1} - 1)j.$$