12.3: The cross product of vectors

Determinants

The determinant of order 2 is given by
\[
\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.
\]

The determinant of order 3 is given by:
\[
\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = a_1 b_1 c_1 - a_2 b_2 c_2 + a_3 b_3 c_3.
\]

The second term has a negative sign since 213 is an odd permutation of 123.

Cross product

Given two vectors \( \mathbf{a} = (a_1, a_2, a_3) \) and \( \mathbf{b} = (b_1, b_2, b_3) \), the cross product of them is a vector given by
\[
\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.
\]

Comment: In the index notation: \((\mathbf{a} \times \mathbf{b})_k = \epsilon_{ijk} a_i b_j\) where \(\epsilon_{ijk} = 1\) if \((ijk)\) is an even permutation of \((1,2,3)\), \(-1\) if it's an odd permutation, and \(0\) otherwise. Repeated indices imply summation over them.

Example 1: Suppose \( \mathbf{a} = (2, -1, 2) \) and \( \mathbf{b} = (2, 2, -1) \). Compute \( \mathbf{a} \times \mathbf{b} \).

Geometric interpretation

To get the interpretation, there are three properties to verify:

- \( \mathbf{a} \times \mathbf{b} \) is perpendicular with both \( \mathbf{a} \) and \( \mathbf{b} \)
- \( |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \).
• The right hand rule holds.

To verify the first, check \((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0\) and \((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0\). To check the second, use the components form. The third one is not so clear. You can gain confidence by doing some simple examples. For example: \(\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}\). (Check them yourself after lecture.)

The second property implies that

\[ |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta \]

Some direct implications:

• If \(\mathbf{a} \times \mathbf{b} = \mathbf{0}\) ⇔ \(\mathbf{a} \parallel \mathbf{b}\)

• \(|\mathbf{a} \times \mathbf{b}|\) is the area of the parallelogram formed by the two vectors.

What happens if \(\mathbf{a} = \mathbf{0}\)?

Example 2: Find the area of the triangle with vertices \(P(1, 3, -2),\)

\(Q(2, 4, 5),\) \(R(-3, -2, 2)\).

Solution. \(S = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| \) □

Mind twister: The area of the parallelogram determined by \(\mathbf{v}\) and \(\mathbf{w}\) is 5. Both \(\mathbf{v}\) and \(\mathbf{w}\) are perpendicular with \((1, 2, 2)\). Compute

\[
\det \begin{pmatrix}
v_1 & v_2 & v_3 \\
w_1 & w_2 & w_3 \\
1 & -1 & 2
\end{pmatrix}
\]

Properties of cross product

• \(\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}\)

• \(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\)

• \(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\)

All equalities can be verified by using the component forms. Verify them after lecture.

Exercise: Simplify \((\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})\).
Some comments:

- Cross product does NOT commute. You should remember this.
- The second one is \( \text{det}(a, b, c) \). The identity holds is therefore natural.
- The third one is very important in engineering applications (not in our course). This is especially useful when combined with differential operators. One example could be \( \nabla \times (\nabla \times E) = \nabla (\nabla \cdot E) - (\nabla \cdot \nabla)E \). What they mean will be explained very later, not here.
- For those who are interested, you can use index notations to derive them: \((b \times c)_k = \epsilon_{ijk} b_i c_j\) and \(a \times (b \times c)_n = \epsilon_{pkn} a_p \epsilon_{ijk} b_i c_j\). Using the identity \(\epsilon_{pkn} \epsilon_{ijk} = \epsilon_{npk} \epsilon_{ijk} = \delta_{in} \delta_{pj} - \delta_{ip} \delta_{jn}\) yields the result.

Scalar Triple product

\[ a \cdot (b \times c) = (a \times b) \cdot c. \] This is a real number, called the scalar triple product.

\[
\begin{vmatrix}
 a_1 & a_2 & a_3 \\
 b_1 & b_2 & b_3 \\
 c_1 & c_2 & c_3 \\
\end{vmatrix}
\]

Geometric meaning

Consider the parallelepiped determined by \(a, b, c\).

Note that \(|b \times c| = |b||c| \sin \theta\) is the area of the parallelogram determined by \(b\) and \(c\), where \(\theta\) is the angle between \(b\) and \(c\).

\[ |a \cdot (b \times c)| = |b \times c| |a| \cos \beta, \] where \(\beta\) is the angle between \(a\) and \(b \times c\).

\[ |a| \cos \beta \] is equal to the height of the parallelepiped for base \(bc\).

Exercise: Use the geometric meaning to explain why \(|a \cdot (b \times c)| = |(a \times b) \cdot c|\).

Example: Let \(A(1, -1, 2), B(2, 0, 1), C(3, 2, 0)\) and \(D(5, 4, z)\).

1. Compute the volume of the parallelepiped determined by \(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\).
2. What happens if \(z = -2\)?
3. If \(z = 1\), figure out the distance from \(D\) to the plane \(ABC\).
Solution. \[ V = |\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})| = |z + 2|. \]

If \( z = -2 \), they are coplanar (in one common plane).
\[ d = V/|\overrightarrow{AB} \times \overrightarrow{AC}|. \]

\[ \Box \]

12.4: Lines and planes in space

Geometric objects such as lines and surfaces can be represented by their position vectors (recall the starting point of a position vector is the origin).

Goal: Find the equations that the position vector or the terminal point \( P(x, y, z) \) satisfies.

Lines

Given a point on the line \( P_0(x_0, y_0, z_0) \) and a vector \( v = (a, b, c) \) that is parallel with the line, we determine that any point on the line \( P(x, y, z) \) satisfies the relation \( \overrightarrow{P_0P} = tv \). The position vector of \( P \) is given by

\[ r = \overrightarrow{OP_0} + tv. \]

This is called the \textbf{vector equation} of the line.