15.4 Green’s theorem

A simple closed curve in plane is one curve \( C, r(t) : t \in [a,b] \) such that \( r(a) = r(b) \), and there are no other intersections.

The positive orientation is counterclockwise.

The first version of Green’s theorem:

**Theorem 1.** If \( C \) is a simple closed curve, positively oriented (i.e. counterclockwise oriented) and the region enclosed by it is \( R \), then for any two continuously differentiable functions \( P(x,y) \) and \( Q(x,y) \), we have

\[
\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA.
\]

The integral can be written as \( \oint F \cdot dr \), where \( F = \langle P, Q \rangle \). This is the work done on the closed loop.

**Example:** Let \( F = \langle 3xy, 2x^2 \rangle \). \( C \) is the boundary of the region bounded by \( y = x \) and \( y = x^2 - 2x \), oriented counterclockwise. Evaluate the work done by \( F \) along \( C \) in two ways.

**Solution.** Way 1: we apply Green’s theorem:

\[
W = \oint_C F \cdot dr = \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA.
\]

In our case, the two curves intersect at \((0, 0)\) and \((3, 3)\). The region can be written as \( 0 \leq x \leq 3, x^2 - 2x \leq y \leq x \). Further,

\[
Q_x = (2x^2)_x = 4x, \quad P_y = (3xy)_y = 3x.
\]

Hence,

\[
W = \int_0^3 \int_{x^2-2x}^x (4x - 3x)\,dx = \int_0^3 x(3x - x^2)\,dx = 27 - \frac{81}{4} = \frac{27}{4}.
\]

Way 2 is to integrate the line integral directly. We see \( C = C_1 + C_2 \).

\( C_1 \) is the parabola. It can be parametrized as

\[
r(t) = \langle t, t^2 - 2t \rangle, 0 \leq t \leq 3.
\]
Hence,

\[ W_1 = \int_{C_1} Pdx + Qdy = \int_0^3 3t(t^2 - 2t)dt + 2t^2(2t - 2)dt \]
\[ = \int_0^3 (7t^3 - 10t^2)dt = \frac{7}{4}t^4 - \frac{10}{3}t^3 |_0^3 = \frac{7 \times 81}{4} - 90 \]

Let’s now look at the second curve. \( C_2 \) is the line segment. It can be parametrized as

\[ r(t) = (3 - 3t, 3 - 3t), 0 \leq t \leq 1. \]

Then, we have

\[ W_2 = \int_{C_2} Pdx + Qdy = \int_0^1 3(3 - 3t)(3 - 3t)(-3dt) + 2(3 - 3t)^2(-3dt) \]
\[ = \int_0^1 (-15) \times 9(1 - t)^2dt = 5 \times 9(1 - t)^3 |_0^1 = -45. \]

The total work is

\[ W = W_1 + W_2 = \frac{7 \times 81}{4} - 90 - 45 = \frac{567 - 4 \times 135}{4} = \frac{27}{4}. \]

We get the same answer using the two ways! \( \square \)