Example: Set up the integral for the area inside the two circles $r = 1$ and $r = 2 \sin \theta$. Set up the integral for the volume of the solid bounded by $r = 1, r = 2 \sin \theta$ and $z = y$.

Solution. For the area, $A = \iint_R dA$. In polar, $dA = rdr \theta$. We see that we must divide the integral into three pieces. $1 = 2 \sin \theta$. We find $\theta = \pi/6$ and $\theta = 5\pi/6$. Hence,

$$A = \int_0^{\pi/6} \int_0^{2 \sin \theta} rdr \theta + \int_{\pi/6}^{5\pi/6} \int_0^1 rdr \theta + \int_{5\pi/6}^{\pi} \int_0^{2 \sin \theta} rdr \theta.$$

The volume is $V = \iiint_R (z_2 - z_1) dA = \iint_R y \, dA$. Hence,

$$V = \int_0^{\pi/6} \int_0^{2 \sin \theta} r \sin \theta \, rdr \theta + \int_{\pi/6}^{5\pi/6} \int_0^1 r \sin \theta \, rdr \theta + \int_{5\pi/6}^{\pi} \int_0^{2 \sin \theta} r \sin \theta \, rdr \theta.$$

Example: Set up the integral for the volume bounded by $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 - 2x = 0$.

Solution. We use cylindrical coordinates. $r^2 + z^2 = 4$ and $r - 2 \cos \theta = 0$. The region is determined by $r - 2 \cos \theta = 0$. We set $r = 0$ and have $\cos \theta = 0$. Hence, $-\pi/2 \leq \theta \leq \pi/2$. Then, $0 \leq r \leq 2 \cos \theta$. For $z$, we find $-\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}$. Note $dV = rdr \theta dz$. The volume is therefore:

$$V = \iiint_T dV = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r dz \, rdr \theta.$$

Example: Set up the integral for the mass of the region contained in the sphere $x^2 + y^2 + (z - a)^2 = a^2$ but below $z = r$ with unit density.

If we draw the picture, we see that the spherical coordinates are the best.
Solution. $z = r$ is just $\phi = \pi/4$. The sphere is $\rho^2 - 2a \rho \cos \phi = 0$ or $\rho = 2a \cos \phi$. Hence, we have $\pi/4 \leq \phi \leq \pi/2, 0 \leq \rho \leq 2a \cos \phi, 0 \leq \theta < 2\pi$. Then,

$$m = \int_{\pi/4}^{\pi/2} \int_0^{2a \cos \phi} \int_0^{2\pi} \rho^2 \sin \phi d\theta d\rho d\phi.$$

□

Example: Consider the ice-cream cone above $\phi = \pi/6$ but below $\rho = 2a \cos \phi$. Suppose the density is $\delta = 1$. Set up the integrals for the total mass and centroid.

Solution. This problem is convenient in spherical coordinates. $0 \leq \phi \leq \pi/6, 0 \leq \theta < 2\pi, 0 \leq \rho \leq 2a \cos \phi$. The volume element is $dV = \rho^2 \sin \phi d\rho d\theta d\phi$.

The total mass is

$$m = \int \int \int_{T} \delta dV = \int_0^{\pi/6} \int_0^{2\pi} \int_0^{2a \cos \phi} 1 \rho^2 \sin \phi d\rho d\theta d\phi.$$

For the centroid, we use symmetry do conclude that $\bar{x} = \bar{y} = 0$. Then,

$$\bar{z} = \frac{1}{m} \int \int \int_{T} \delta z dV = \frac{1}{m} \int_0^{\pi/6} \int_0^{2\pi} \int_0^{2a \cos \phi} \rho \cos \phi \rho^2 \sin \phi d\rho d\theta d\phi$$

because $z = \rho \cos \phi$. □