Math 212-Lecture 20

14.4 Double integrals in polar coordinates

We consider the amplification factor for the transformation \( x = r \cos \theta, \) \( y = r \sin \theta. \) The Jacobian is

\[
J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r.
\]

This tells us that \( dA = rdrd\theta \) (where \( dA \) is the area element in \( xy \) plane.)

By the picture (show this in class), we can determine that the area is \( dA = rdrd\theta \) directly. Hence, we can do \( dx dy \to rdrd\theta. \)

Example: Evaluate the integral:

\[
\int_0^2 \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{-x^2-y^2} dy dx.
\]

Solution. The region is \( 0 \leq x \leq 2 \) and \(-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}. \) This is the right half of the disk centered at \((0,0)\) with radius 2.

We use the polar coordinates. \( 0 \leq r \leq 2 \) and \(-\pi/2 \leq \theta \leq \pi/2. \) \( dy dx \to rdrd\theta. \) We have

\[
\int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} r dr d\theta = \pi \int_0^2 r e^{-r^2} dr = -\frac{\pi}{2} e^{-r^2} \bigg|_0^2 = \frac{\pi}{2} (1 - e^{-4}).
\]

Polar coordinates is convenient for radially simple region \( \alpha \leq \theta \leq \beta, r_1(\theta) \leq r \leq r_2(\theta). \)

Example: Let’s evaluate the volume of the solid bounded by \( z = x^2 + y^2 \) and \( z = y. \)

Solution. Previously (see the lecture notes for volumes using double integrals), we agreed that the volume is

\[
V = \iint_R (y - x^2 - y^2) dA
\]

where \( R \) is the region bounded by the circle \( x^2 + y^2 = y. \) This equation is \( r = \sin \theta \) in polar coordinates.
Letting \( r = 0 \), we have \( \theta = 0, \pi \) (note that \( 0 \to 2\pi \) will cover the disk twice.) hence, the region is \( 0 \leq \theta \leq \pi, 0 \leq r \leq \sin \theta \). The integral becomes

\[
\int_0^\pi \int_0^{\sin \theta} (r \sin \theta - r^2) r dr d\theta = \int_0^\pi \frac{1}{12} \sin^4 \theta d\theta.
\]

We are integrating even powers of \( \sin \theta \). We do \( \sin^2 \theta = (1 - \cos(2\theta))/2 \).

Then,

\[
\frac{1}{12} \int_0^\pi \frac{1}{4} [1 + \cos^2(2\theta)] d\theta,
\]

since the integral of \(-2 \cos 2\theta \) is zero. Lastly, \( \cos^2(2\theta) = (1 + \cos(4\theta))/2 \).

The final answer is \( \pi/32 \). \( \square \)

**Example:** Evaluate \( I = \int_0^\infty e^{-x^2} dx \).

**Solution.** \( I = \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-y^2} dy \). hence,

\[
I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_R e^{-x^2-y^2} dA,
\]

where \( R \) is the first quadrant. In polar, \( 0 \leq r < \infty \) and \( 0 \leq \theta \leq \pi/2 \). Then,

\[
I^2 = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta = \frac{\pi}{4}.
\]

hence, \( I = \sqrt{\pi}/2 \). \( \square \)

**Exercise.** Set up the integral for the volume under \( f(x, y) = x^2 \) and above the region \( D = \{(x, y) : x^2 + y^2 \leq 4, x^2 + (y - 2)^2 \leq 4 \} \) using polar coordinates.

### 14.6 Triple integrals

Suppose \( T \) is a region in 3D space, contained in the domain of \( f(x, y, z) \). The triple integral of \( f \) over \( T \) is defined by the limit of Riemann sums:

\[
\iiint_T f(x, y, z) dV = \lim_{|P| \to 0} \sum_i f(x^*_i, y^*_i, z^*_i) \Delta V.
\]

Like the double integrals, we can change the triple integrals to iterated integrals. (Think about the Riemann sums. There is a 3D array of blocks. We can group the blocks in different ways. In principle, we have 6 ways. \( dx dy dz, dxdzdy, dydxdz \) etc.)
For a solid $T$, the volume is
\[ V = \iiint_T dV. \]

For a 3D region, given density $\delta$ (mass per unit volume), we find similarly the total mass
\[ m = \iiint_T \delta dV, \]
the centroid $(\bar{x}, \bar{y}, \bar{z})$, where
\[ \bar{x} = \frac{1}{m} \iiint_T x \delta dV. \]
and moments of inertia. Note that the moments of inertia would be slightly different. For example, $I_x$. The distance to $x$-axis is $\sqrt{y^2 + z^2}$ instead of $|y|$. Hence, we have
\[ I_x = \iiint_T (y^2 + z^2) \delta(x, y, z) dV. \]

A $z$-simple region means every vertical line intersects the region with a single line segment. Then, for given $x, y$ the limits for $z$ would depend on $x, y$. We define $x$-simple or $y$-simple regions similarly.

**Example:** Set up the integral for the volume of the solid bounded by $y + z = 4, y = 4 - x^2, y = 0, z = 0$.

**Solution.** In double integral way, this is
\[ V = \iint_{R} (z_{\text{high}} - z_{\text{low}}) dA = \iint_{R} (4 - y - 0) dA. \]
continue with writing the region $R$

In triple integral way,
\[ V = \iiint_{T} dV. \]
Let’s figure out $T$. The projection of the solid onto $xy$ plane is the one determined by $y = 4 - x^2$ and $y = 0$. Hence, we have $-2 \leq x \leq 2, 0 \leq y \leq 4 - x^2$. Then clearly, for given $(x, y)$, $0 \leq z \leq 4 - y$. Hence, the volume is given by
\[ V = \int_{-2}^{2} \int_{0}^{4-x^2} \int_{0}^{4-y} dz dy dx = \int_{-2}^{2} \int_{0}^{4-x^2} (4 - y - 0) dy dx. \]
This means if we integrate $z$ coordinate first, the triple integral can reduce to the double integral way exactly. However, the triple integral is more general as sometimes we integral $x$ first or so.

Example: Find the mass of the pyramid $T$ with vertices $(0, 0, 0), (0, 3, 0), (2, 0, 0)$ and $(0, 0, 6)$, if the density is given by $\delta = z$.

For the plane, you can find the normal vector first using cross product. However, in this special case, we can construct it quickly: if a plane intersects with $x, y, z$ at $(a, 0, 0), (0, b, 0), (0, 0, c)$ respectively, then the equation is $x/a + y/b + z/c = 1$. 