Math 212-Lecture 2

Topics: Dot product;

12-1: continue’d

• Triangle rule indicates: \( \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \ldots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{A_1A_n} \)

• Can move the vectors freely as long as the directed line segment carries the same magnitude and direction. To be convenient, we choose the starting point to be the origin \( O \). Then, vector \( \overrightarrow{OP} \) describes the position of \( P \), called the position vector.

• Given \( Q(a_1, b_1) \), \( R(a_2, b_2) \), \( \overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \langle a_2 - a_1, b_2 - b_1 \rangle \).

12-2: Three-dimensional vectors

Cartesian/Rectangular coordinates in space

• We need three coordinate axes: \( x, y, z \). Right-handed.

• Coordinate planes: \( xy, yz, xz \) planes. Octants.

• Any point \( P \) has a unique rectangular coordinate \( P(x, y, z) \).

• Given two points \( P_1(x_1, y_1, z_1) \) and \( P_2(x_2, y_2, z_2) \), the distance between them:

\[
d = |P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}. \text{(Pythagorean again.)}
\]

Midpoint \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right) \).

Study the sphere yourself!

Vectors in space

Similar as the vectors in plane. Given \( i, j, k \), the basic unit vectors along the three axes, \( v = xi + yj + zk \). Then, we identify

\[
v = \langle x, y, z \rangle.
\]

Length(magnitude), addition, scalar multiplication are similar.
Dot product

\[ \mathbf{a} = a_1 \mathbf{i} + b_1 \mathbf{j} + c_1 \mathbf{k} = \langle a_1, b_1, c_1 \rangle, \quad \mathbf{b} = a_2 \mathbf{i} + b_2 \mathbf{j} + c_2 \mathbf{k} = \langle a_2, b_2, c_2 \rangle. \]

The dot product is defined to be a number (scalar), given by

\[ \mathbf{a} \cdot \mathbf{b} = a_1 a_2 + b_1 b_2 + c_1 c_2. \]

Properties:

- \( \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \)
- \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \)
- \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \)

We show the first as an example:

**Proof.** \( \mathbf{a} \cdot \mathbf{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 = (\sqrt{a_1^2 + a_2^2 + a_3^2})^2 = |\mathbf{a}|^2. \)

*Exercise: Show the distribution law holds.*

**Example 1:** Suppose \(|\mathbf{a}| = 2, |\mathbf{b}| = 3\) and \(\mathbf{a} \cdot \mathbf{b} = -4\). Find \(|\mathbf{a} + 2\mathbf{b}|\).

**Solution.** We use \(|\mathbf{a} + 2\mathbf{b}| = \sqrt{(\mathbf{a} + 2\mathbf{b}) \cdot (\mathbf{a} + 2\mathbf{b})}\) and distribute out the dot products.

Interpretation of dot product

Let \( \mathbf{a} = \overrightarrow{OP} \) and \( \mathbf{b} = \overrightarrow{OQ}. \) (In other words, we move the two vectors such that they both start from the origin. Then, their terminal points are given by \( P \) and \( Q \) respectively.) The angle between the two vectors is \( \theta = \angle POQ. \)

**Theorem 1.** Let \( \theta \) be the angle between \( \mathbf{a} \) and \( \mathbf{b} \). Then,

\[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta. \]

The proof follows directly from law of cosines if \( \mathbf{a} \neq 0, \mathbf{b} \neq 0. \) Read the details.

The following formula is used frequently:

\[ \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}. \]

- If both \( \mathbf{a} \) and \( \mathbf{b} \) are not zero vector, \( \mathbf{a} \perp \mathbf{b} \iff \theta = \frac{\pi}{2} \iff \mathbf{a} \cdot \mathbf{b} = 0. \)
- If \( \mathbf{a} \neq 0, \mathbf{b} \parallel \mathbf{a} \iff \mathbf{b} = \lambda \mathbf{a} \) for some \( \lambda \in \mathbb{R}. \)

*Exercise: what if \( \mathbf{a} = 0? \)*

**Example 2:** Suppose \( A(1,0,0), B(1,2,0), C(-1,-2,-2). \) Find the angle \( \angle BAC. \)
Direction angles and direction cosines

The direction angles of \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) are the angles between \( \mathbf{a} \) and \( x \)-axis (\( y \)-axis, or \( z \)-axis). The three angles are denoted by \( \alpha, \beta, \gamma \). \( \cos \alpha, \cos \beta, \cos \gamma \) are called the direction cosines.

\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.
\]

For example: \( \cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{||\mathbf{a}||} = \frac{a_1}{||\mathbf{a}||} \).

Projections

Suppose \( \mathbf{b} \neq 0 \). Component of \( \mathbf{a} \) along \( \mathbf{b} \) is the ‘signed’ length of the perpendicular projection of \( \mathbf{a} \) onto \( \mathbf{b} \). This is a number

\[
\text{comp}_\mathbf{b} \mathbf{a} = |\mathbf{a}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||}.
\]

We can decompose \( \mathbf{a} \) into two parts

\[
\mathbf{a}_\parallel = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||} \right) \frac{\mathbf{b}}{||\mathbf{b}||} = \frac{(\mathbf{a} \cdot \mathbf{b}) \mathbf{b}}{||\mathbf{b}||^2}
\]

and

\[
\mathbf{a}_\perp = \mathbf{a} - \mathbf{a}_\parallel.
\]

**Exercise:** From the picture, it’s easily known that \( \mathbf{a}_\perp \perp \mathbf{b} \). Please verify this.

**Example 3:** Given \( \mathbf{a} = \langle 1, -1, 2 \rangle \), \( \mathbf{b} = \langle 2, 2, 1 \rangle \). Find the smallest value of \( |\mathbf{a} - \lambda \mathbf{b}| \) if \( \lambda \) varies. What is the corresponding \( \lambda \)?

**Solution.** By the picture, we must have \( \mathbf{a} - \lambda \mathbf{b} = \mathbf{a}_\perp \) when it gets the smallest value. \( \square \)

Work \( W = \mathbf{F} \cdot \mathbf{D} = \mathbf{F}_\parallel \cdot \mathbf{D} \).