Math 212-Lecture 10

13.6: Increments and linear approximation

In the previous lecture, we see that the existence of partial derivatives $f_x, f_y$ doesn’t imply that the function is continuous. However, if $f_x$ and $f_y$ are continuous at point $(a, b)$, then $f$ is continuous at $(a, b)$ and furthermore, we have the estimate for $f(a + \Delta x, b + \Delta y) - f(a, b)$:

Consider the increment $\Delta f = f(a + \Delta x, b + \Delta y) - f(a, b)$. We have by the mean value theorem:

$$\Delta f = [f(a + \Delta x, b + \Delta y) - f(a + \Delta x, b)] + [f(a + \Delta x, b) - f(a, b)] = f_y(a + \Delta x, b + \xi(\Delta y)) \Delta y + f_x(a + \eta(\Delta x), b) \Delta x$$

Since $f_x, f_y$ are continuous at $(a, b)$, $|f_y(a + \Delta x, b + \xi(\Delta y)) - f_y(a, b)| \to 0$ and $|f_x(a + \eta(\Delta x), b) - f_x(a, b)| \to 0$. Hence,

$$\Delta f = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \text{small}.$$ 

From this expression, $f$ is continuous at $(a, b)$ and we have the linear approximation of $f(x, y)$ near $(a, b)$:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

This is good if $|x - a|$ and $|y - b|$ are small.

Example: Compute $e^{0.1 \sqrt{17}}$ approximately.

Solution. Since $\sqrt{17} = 4\sqrt{1 + 1/16}$. We can therefore define a function

$$f(x, y) = 4e^{x\sqrt{1+y}}.$$ 

We are supposed to compute $f(0.1, 1/16)$.

$$f(0, 0) = 4$$
$$f_x(x, y) = 4e^{x\sqrt{1+y}} \Rightarrow f_x(0, 0) = 4$$
$$f_y(x, y) = 2e^{x \frac{1}{\sqrt{1+y}}} \Rightarrow f_y(0, 0) = 2.$$ 

Hence,

$$f(x, y) \approx 4 + 4 * 0.1 + 2 \frac{1}{16} = 4.525$$
The differential is thus
\[ df = f_x dx + f_y dy = \langle dx, dy \rangle \cdot \langle f_x, f_y \rangle = dr \cdot \nabla f. \]

where \( \nabla f = \langle f_x, f_y \rangle \) is called the gradient vector. \( \langle dx, dy \rangle = dr \) is the differential of the position vector.

We have the product rule and chain rule:
\[
\begin{align*}
  d(fg) &= g df + f dg \\
  d(f(h)) &= f'(h) dh.
\end{align*}
\]

Example: Find the differential for \( w = x \tan(yz) \).

Solution.
\[
\begin{align*}
  dw &= \tan(yz)dx + x d(\tan(yz)) = \tan(yz)dx + x \sec^2(yz)d(yz) \\
   &= \tan(yz)dx + \sec^2(yz)(ydz + zdy).
\end{align*}
\]

Intuitively, differential implies how the infinitesimal change of the function value relies on the infinitesimal changes of the independent variables. This is from the linear approximation. However, as long as we have this differential expression, it is actually a linear mapping between \( dx, dy \) and \( df \). In rigorous mathematical view, the differential is a mapping from the tangent space of the independent variables to the tangent space of the dependent variable, and \( dx, dy \) don’t even have to be small.

Tangent planes and tangent lines as graphs of linear approximations

The tangent plane of \( z = f(x, y) \) at \( (a, b, f(a, b)) \) is given by
\[
  z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),
\]
which is exactly the graph of the linear approximation.

We know the level set \( f(x, y) - f(a, b) = 0 \) is a curve in \( xy \) plane. Then, the tangent line of the curve at \( (a, b) \) is also given by the graph of the linear approximation:
\[
  f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) - f(a, b) = 0.
\]
or

\[ f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0. \]

**Example:** Consider \( F(x, y, z) = \ln(x^2 + y^2 + z^2) \). The level set \( F(x, y, z) = \ln(3) \) is a surface in 3D space passing through \((1, 1, 1)\). Find the tangent plane of this surface at \((1, 1, 1)\).

The idea is to use linear approximation. \( F(x, y, z) - F(1, 1, 1) \approx F(1, 1, 1) + F_x(1, 1, 1)(x - 1) + F_y(1, 1, 1)(y - 1) + F_z(1, 1, 1)(z - 1) - F(1, 1, 1) = 0 \), or

\[ F_x(1, 1, 1)(x - 1) + F_y(1, 1, 1)(y - 1) + F_z(1, 1, 1)(z - 1) = 0 \]

**Solution.**

\[
\begin{align*}
F_x(x, y, z) &= \frac{2x}{x^2 + y^2 + z^2} \Rightarrow F_x(1, 1, 1) = \frac{2}{3} \\
F_y(x, y, z) &= \frac{2y}{x^2 + y^2 + z^2} \Rightarrow F_y(1, 1, 1) = \frac{2}{3} \\
F_z(x, y, z) &= \frac{2z}{x^2 + y^2 + z^2} \Rightarrow F_y(1, 1, 1) = \frac{2}{3}.
\end{align*}
\]

The tangent plane is hence

\[
\frac{2}{3}(x - 1) + \frac{2}{3}(y - 1) + \frac{2}{3}(z - 1) = 0.
\]

\[ \square \]

**Differentiability**

Let \( x = \langle x, y \rangle \) or \( x = \langle x, y, z \rangle \) and \( h = \langle h_1, h_2 \rangle \) or \( h = \langle h_1, h_2, h_2 \rangle \). \( f \) is said to be differentiable at \( x \), if there exists a vector \( c \) such that

\[
\lim_{|h| \to 0} \frac{f(x + h) - f(x) - c \cdot h}{|h|} = 0.
\]

Facts: 1. If \( f_x, f_y, f_z \) are continuous (In this case, \( f \) is called continuously differentiable), \( c = \nabla f \) evaluated at \( x \) works by the linear approximation. Hence, the function is differentiable.

2. Differentiability implies that partial derivatives exist and the function is continuous at that point \( x \). Further, \( c = \nabla f \). (Recall that the existence of partial derivatives doesn’t imply the continuity.) However, the partial derivatives may not be continuous at the point.

We see that the existence of partial derivatives doesn’t imply the differentiability (It doesn’t even imply the continuity of the function), but continuity of the partial derivatives is something stronger than the differentiability.