My research interest lies broadly in applied mathematics, including mathematical physics, numerical analysis and stochastic analysis. Since 2015 when I started my postdoc position (Elliott Assistant Research Professor) at Duke University, I focused my research on the following subjects mainly being mentored by Prof. Jian-Guo Liu:

1. A generalized definition of time fractional Caputo derivative, and some basic theories in time fractional ODEs, SDEs, and PDEs. In particular, a fractional SDE model satisfying fluctuation-dissipation theorem and some compactness criteria for weak solutions of time fractional PDEs have been proposed.

2. SDE related problems, including a Gaussian mixture method for solving SDEs numerically and SDE approximation for stochastic gradient descent method or online PCA.

3. Numerical analysis for water wave related problems.

4. Analysis of some PDE models including modified Camassa-Holm equations and p-Navier-Stokes equations.

Currently, I am focusing on topics related to optimal transport and machine learning, and some continuation of the time fractional differential equations. Below, let me introduce the above accomplishments in more details and summarize my future plans.

1 Summary of my postdoc research

In this section, I will summarize my research during postdoc in some details. Most of the projects listed here have been published or submitted, while others are near completion. The collaborators for these projects include Jian-Guo Liu, Jianfeng Lu (Duke University), Yu Gao (Harbin Institute of Technology), Robert Pego (Carnegie Mellon University), Xiaoqian Xu (Carnegie Mellon University), Zhennan Zhou (Peking University) to name a few. Besides the projects listed here, there are several on-going projects which I choose to omit.

1.1 Time fractional differential equations

Fractional calculus [1] has been used to model the ubiquitous memory effects in physics and engineering, e.g. particles in heat bath, soft matter with viscoelasticity. The Caputo's definition of fractional derivatives was first introduced in [2] to study the memory effect of energy dissipation, and soon became a useful modeling tool to construct physical models for nonlocal interactions in time. Recently, modeling using fractional partial differential equations with Caputo derivatives has been justified by certain limiting processes and probability [3, 4].

In [5], Jian-Guo Liu and I extended the definition of Caputo derivatives to a certain class of integral functions based on a convolution group. This definition is more useful theoretically compared with the traditional definition since it reveals the underlying group structure and the derivatives can be defined as long as the functions are locally integrable and the initial value can be defined in some sense. The group structure allows us to convert the differential form $D_c^\gamma \varphi = f$ into the Volterra integral form for $t > 0$

$$\varphi = \varphi(0+) + g_\gamma * (\theta f) = \varphi(0+) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s)\, ds$$

where θ is the standard Heaviside step function. This definition is useful because it can clearly reveal the difference between $D_c^\alpha D_c^\beta$ and $D_c^{\alpha+\beta}$.

Our idea was partially motivated by the book of Gel'fand and Shilov [6]. In [6], the integrals and derivatives of arbitrary order for a distribution $\varphi \in \mathscr{D}'(\mathbb{R})$ supported on $[0, \infty)$ are defined as

$$\varphi^{(\alpha)} = \varphi * g_\alpha, \quad g_\alpha := \frac{t_+^{\alpha-1}}{\Gamma(\alpha)},\ \alpha \in \mathbb{C}.$$

Here $t_+ = \max(t, 0)$ and $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ must be understood as distributions for $\Re\alpha \le 0$. We then found more convenient expressions for the distributions $g_\alpha = \frac{(\theta(t)t)^{\alpha-1}}{\Gamma(\alpha)}$ when $\alpha \le 0$ and then extended the idea to define fractional calculus of a certain class of distributions $\mathscr{E}^T \subset \mathscr{D}'(-\infty, T), T \in (0, \infty]$:

$$\varphi \mapsto I_\alpha \varphi := g_\alpha * \varphi,\ \forall \varphi \in \mathscr{E}^T,\ \alpha \in \mathbb{R}.$$

We then make distributions causal (i.e. zero when $t < 0$) by, roughly, multiplying $\theta(t)$ (see [5] for more rigorous treatment) and then define

$$J_\alpha \varphi := I_\alpha(\theta(t)\varphi(t)),\ \alpha \in \mathbb{R}.$$

These operators form a group, and when acting on causal functions, they agree with the traditional Riemann-Liouville definition for $t > 0$ but include some singularities at $t = 0$ so that the group property holds. Using these modified Riemann-Liouville operators, we define the generalized definition of Caputo derivatives for $\gamma \in (0, 1)$ as

$$D_c^\gamma \varphi := J_{-\gamma}\varphi - \varphi(0+)\frac{\theta(t)}{\Gamma(1-\gamma)} t^{\gamma-1} = J_{-\gamma}(\varphi - \varphi(0+)).$$

Using this new definition in [5], Feng, Liu, Xu and I studied the one dimensional autonomous ODEs in [7]:

$$D_c^{\gamma} u = f(u).$$

The monotonicity and blowup behaviors of the solutions have been discussed thoroughly. Besides, we also studied the discrete equations (numerical schemes) and proved some discrete Grönwall inequalities.

In [8], we extended the definition of Caputo derivatives to weak Caputo derivatives for functions valued in general Banach spaces through an integration by parts formula. Based on a Volterra type integral form, we established some time regularity estimates of the functions provided that the weak Caputo derivatives are in certain spaces. We then established the compactness criteria over space and time with the time regularity estimates, which are analogies of the traditional Aubin-Lions lemma. As well-known, Aubin-Lions lemma is powerful for establishing the existence of weak solutions to nonlinear PDEs, especially of parabolic type. Our compactness criteria can play similar roles for time fractional PDEs. Actually, the existence of weak solutions for a special case of time fractional compressible Navier-Stokes equations with constant density and time fractional Keller-Segel equations in $\mathbb{R}^2$ were then proved as examples.

Another interesting problem I studied is the fractional stochastic differential equations with Jianfeng Lu and Jian-Guo Liu [9]. We argued that for a physical system the Caputo derivative must be used to pair with fractional noise so that the fluctuation-dissipation theorem would be satisfied:

$$D_c^{2-2H} x = -\nabla V(x) + C_H \dot{B}_H.$$

For the linear force regime, we showed the convergence to the Gibbs measure in algebraic rate, and provided some frameworks for studying the nonlinear force regimes.

As a byproduct of the research of the time fractional differential equations, we also proved the stability of de-convolution for one-sided convolution with completely monotone sequences in [10].

1.2 SDE related problems

The first piece of work related to SDE I would like to introduce is the Gaussian mixture method for numerically solving SDEs, which is a joint work with Jianfeng Lu, Jonathan Mattingly (Duke) and Lihan Wang (Duke). The goal is to design a simple and fast numerical scheme to solve

$$dX = b(X)\,dt + \sigma(X)\,dW$$

numerically with high weak order. Our strategy is to approximate the probability measure

$$\mu(X^{n+1}|X^n = x_n)$$

with a mixture of Gaussians. To sample X^{n+1}, we choose one Gaussian randomly and then sample from that Gaussian. The mean and variance of the chosen Gaussian can be solved using simple ODEs. Compared to traditional high order weak schemes, which are usually based on Itô-Taylor expansion or involve solving iterated Itô integrals, our strategy is easy (only need to sample Gaussians) and fast (linear in d (the dimension) for each step).

Another work I would like to mention here is the study of semigroups for the stochastic gradient descent (SGD) and online PCA algorithms from machine learning (with Yuanyuan Feng from CMU and Jian-Guo Liu) [11]. We showed rigorously that stochastic differential equations (SDEs) in $\mathbb{R}^d$ (on the sphere $\mathbb{S}^{d-1}$) can be used to approximate SGD (online PCA) weakly. These SDEs can provide some insights of the behaviors of these algorithms.

1.3 Numerical analysis for waterwave related problems

The interface problems of inviscid fluids, called waterwave problems, are described by the irrotational Euler equations, dating back to Stokes and Taylor. The breakthrough work for the well-posedness of the waterwave problems in Sobolev spaces was done by Wu in [12]. A key tool used by Wu is the conformal mapping formulation. The power of conformal mapping is to change the hard free surface problem into a problem on a fixed simple domain.

Using conformal mapping to numerically solve waterwave problems was discussed in a series of works by Zakharov et al ([13, 14]). However, they never did rigorous numerical analysis of the conformal mapping formulation. With Jian-Guo Liu and Robert Pego, I studied rigorously the convergence of the conformal mapping formulation of the water wave problems for a two dimensional drop. Specifically, we showed that the flow map preserves to be an injection as long as the boundary stays to be a Jordan curve, so that all nonlocal terms can be represented in terms of the Hilbert transform on the circle. The resulted system of equations read ($\Lambda = (-\Delta)^{1/2} = H\partial_\theta$)

$$\begin{aligned}
X_t &= (\Lambda X)\frac{\Lambda\Phi}{J} + X_\theta\left(H\left(\frac{\Lambda\Phi}{J}\right)\right),\\
\Phi_t &= -p - V + \frac{(\Lambda\Phi)^2 - \Phi_\theta^2}{2J} + \Phi_\theta\left(H\left(\frac{\Lambda\Phi}{J}\right)\right),
\end{aligned}$$

with the definitions $J = X_\theta^2 + (\Lambda X)^2$, $p = \gamma J^{-3/2}(X_\theta \,\Lambda X_\theta - X_{\theta\theta}\,\Lambda X)$. We then developed a filtered pseudo-spectral method for these two equations, and used some analysis tools from [15] to show that the pseudo-spectral method converges.

There is a nonlocal hyperbolic system that is closely related to the water wave problems (see [15, 12].):

$$\partial_t \eta = \sigma(\alpha, t)(-\Delta)^{1/2}\zeta + g_1,$$
$$\partial_t \zeta = -c(\alpha, t)\eta + g_2$$

In the constant coefficient case and $g_1 = g_2 = 0$, this nonlocal system then leads the following interesting nonlocal hyperbolic equation:

$$u_{tt} = -(-\Delta)^{1/2}u.$$

Zibu Liu (Peking University), Zhennan Zhou (Peking University), Yi Yang (Tsinghua University) and I started studying the numerical schemes of this nonlocal hyperbolic system since the summer of 2017. The CFL condition for this system is like $\tau/\sqrt{h} < C$ and studying this system is helpful to understand the time discretization in the waterwave problems.

1.4 PDE analysis

Another branch of my research is analysis of some PDE models from mathematical physics.

Yu Gao, Jian-Guo Liu and I studied the modified Camassa-Holm (mCH) equation,

$$m_t + 2ku_x + [(u^2 - u_x^2)m]_x = 0, \quad m = u - u_{xx}, \quad x \in \mathbb{R}, \ t > 0,$$

with cubic nonlinearity. By extending some techniques for Camassa-Holm equations ([16, 17]), we developed a patching technique to change unbounded and multivalued traveling wave solutions into single-valued bounded weak solutions [18]. The traveling wave solutions to the mCH equation are given by the level sets of the following Hamiltonian

$$H = \frac{1}{4}(\phi^2 - v^2)^2 - \frac{1}{2}c(\phi^2 - v^2) + k\phi^2 - g\phi,$$

with c, g being some parameters. The Hamiltonian here is a fourth order polynomial which makes the level sets nontrivial. As a consequence, some level sets correspond to unbounded or multi-valued solutions. We noticed that if we use the hyperbola

$$\phi^2 - \frac{1}{3}v^2 = c$$

to cut arcs from the level set, we can obtain new solutions by jointing these arcs together. This method then generates a family of interesting weak solutions to mCH equation. Another work along this direction is to construct a global peakon weak solutions using a dispersive regularization for the case $k = 0$ [19].

Another PDE model I studied with Jian-Guo Liu is the p-Navier-Stokes equations [20]. This work is a generalization of the classical Euler equations and Navier-Stokes equations. We first of all derived p-Euler equations as the Euler-Lagrange equations for the action represented by the Benamou-Brenier characterization of Wasserstein-p distances, with incompressibility constraint. In other words, we minimize

$$\mathcal{A} = \int_0^1 \int_{\Omega_t} \frac{1}{p}|v|^p \, dxdt$$

with the constraint that $\nabla \cdot v = 0$ so that $|\Omega_t| = |\Omega_0|$. v is the Eulerian velocity field of the flow for $\Omega_t : 0 \le t \le 1$. The derived system of equations for $x \in \Omega_t$ read:

$$\begin{cases} \partial_t v_p + v \cdot \nabla v_p = -\nabla \pi, \\ v_p = |v|^{p-2} v, \\ \nabla \cdot v = 0. \end{cases} \tag{1.1}$$

p-Euler equations have similar structures with the usual Euler equations but the 'momentum' is the signed $(p-1)$-th power of the velocity. The 2D p-Euler equations have a streamfunction-vorticity formulation for $x \in \Omega$:

$$\begin{cases} \partial_t \omega_p + v \cdot \nabla \omega_p = 0, \\ -\Delta_p \psi = -\nabla \cdot (|\nabla \psi|^{p-2} \nabla \psi) = \omega_p, \\ v = \nabla^{\perp} \psi. \end{cases}$$

The vorticity becomes p-Laplacian of the stream function. This system of equations share similarities with the surface quasi-geostrophic equations.

Adding viscous term given by p-Laplacian of velocity, we obtain what we call p-Navier-Stokes (p-NS) equations for $x \in \Omega \subset \mathbb{R}^d$:

$$\begin{cases} \partial_t v_p + v \cdot \nabla v_p = -\nabla \pi + \nu \Delta_p v, \\ v_p = |v|^{p-2} v, \\ \nabla \cdot v = 0. \end{cases}$$

We then showed the existence of weak solutions to this system using energy estimates and a compactness criterion.

2 A brief summary for my Ph.D research

I did my Ph.D. at University of Wisconsin-Madison and worked with Saverio Spagnolie from 2013 to 2015. I focused on using mathematical tools to solve problems in fluid mechanics, and developing numerical methods for specific problems. The topics include sedimentation and swimming in Stokes flows, boundary integral methods for stokes flows and level set method for Eulerian immersed boundary method and immersed interface method. For the details, one may refer to [21, 22, 23, 24, 25, 26].

3 Future plans

As already stated, I am interested broadly in applied math, and happy to work on any interesting problems arising in the interdisplinary areas. However, my near future research plans could be summarized as following:

1. Continuation of the research in time fractional differential equations.

   Within our new framework for the time fractional differential equations, I would like to analyze the asymptotic behaviors of the solutions to some time fractional differential equations, and explore the roles of memory. For example, the behaviors of solutions the time fractional conservation laws $D_c^{\gamma}u + f(u)_x = 0$ and its possible application to to absorbing boundary conditions for wave equations. Indeed, I have started thinking about with Jian-Guo Liu, Zhennan Zhou and Zibu Liu. Another problem is to explore the role of velocity field to the time fractional advection-diffusion equations $D_c^{\gamma}u + v(x)\cdot\nabla u = \Delta u$. I would like to see whether the velocity field has any diffusion enhancing effects.

2. Numerical algorithms for SDEs and PDEs.

   The Gaussian mixture approximation is a new numerical idea for solving and sampling SDEs. I would like to explore it further. For example, it could be possibly used to get the approximation of sample paths from a potential well to another well. Also, I would like to see whether there is deeper insight of the approximation so that this method can be generalized to more accurate algorithms.

3. One direction I just got interested in is to apply stochastic analysis, optimal transport theories to some algorithms from sampling and machine learning.

As soon as we relate SDEs with small temperature to machine learning algorithms, we can use the viewpoint in SDEs or control problems to design algorithms in machine learning. This is one direction I want to explore. Further, recently, there are many interesting algorithms for optimal transport. For example, a recent work by Leger is [27], where the regularization makes some approximation of optimal transport algorithm fast and efficient. Another example is the W-GAN algorithm [28] for applying the theory of optimal transport to machine learning. I would like to explore these subjects and figure out some math problems.

References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*, volume 204. 2006.

[2] M. Caputo. Linear models of dissipation whose Q is almost frequency independent–II. *Geophysical Journal International*, 13(5):529–539, 1967.

[3] M. Hairer, G. Iyer, L. Koralov, A. Novikov, and Z. Pajor-Gyulai. A fractional kinetic process describing the intermediate time behaviour of cellular flows. *The Annals of Probability*, In press.

[4] Z.-Q. Chen. Time fractional equations and probabilistic representation. *Chaos, Solitons & Fractals*, 2017.

[5] L. Li and J.-G. Liu. A generalized definition of Caputo derivatives and its application to fractional odes. *arXiv preprint arXiv:1612.05103v2*, 2017.

[6] I. M. Gel'fand and G. E. Shilov. *Generalized functions*, volume 1. Academic Press, 1964.

[7] Y. Feng, L. Li, J.-G. Liu, and X. Xu. Continuous and discrete one dimensional autonomous fractional odes. *Discrete Contin. Dyn. Syst. Ser. B*, To appear.

[8] L. Li and J.-G. Liu. Some compactness criteria for weak solutions of time fractional PDEs. *arXiv preprint arXiv:1708.08384*, 2017.

[9] L. Li, J.-G. Liu, and J. Lu. Fractional stochastic differential equations satisfying fluctuation-dissipation theorem. *J. Stat. Phys.*, 169:316–339, 2017.

[10] L. Li and J.-G. Liu. A note on deconvolution with completely monotone sequences and discrete fractional calculus. *Q. Appl. Math.*, 2017.

[11] Y. Feng, L. Li, and J.-G. Liu. A note on semi-groups of stochastic gradient descent and online principal component analysis. *preprint*.

[12] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-d. *Invent. Math.*, 130(1):39–72, 1997.

[13] V.E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phy.*, 9(2):190–194, 1968.

[14] A.I. Dyachenko, E.A. Kuznetsov, M.D. Spector, and V.E. Zakharov. Analytical description of the free surface dynamics of an ideal fluid (canonical formalism and conformal mapping). *Phys. Lett. A*, 221(1):73–79, 1996.

[15] J.T. Beale, T.Y. Hou, and J.S. Lowengrub. Convergence of a boundary integral method for water waves. *SIAM J. Numer. Anal.*, 33(5):1797–1843, 1996.

[16] Z. Liu, R. Wang, and Z. Jing. Peaked wave solutions of Camassa–Holm equation. *Chaos, Solitons Fract.*, 19(1):77–92, 2004.

[17] J. Lenells. Traveling wave solutions of the Camassa–Holm equation. *J. Differential Equations*, 217(2):393–430, 2005.

[18] Y. Gao, L. Li, and J.-G. Liu. Patched peakon weak solutions of the modified Camassa-Holm equation. *arXiv preprint arXiv:1703.07466*, 2017.

[19] Y. Gao, L. Li, and J.-G. Liu. A dispersive regularization for the modified Camassa-Holm equation. *arXiv preprint arXiv:1707.06377*, 2017.

[20] L. Li and J.-G. Liu. p-Euler equations and p-Navier-Stokes equations. *arXiv preprint arXiv:1706.05693*, 2017.

[21] L. Li, H. Manikantan, D. Saintillan, and S. E. Spagnolie. The sedimentation of flexible filaments. *J. Fluid Mech.*, 735:705–736, 2013.

[22] H. Manikantan, L. Li, S. E. Spagnolie, and D. Saintillan. The instability of a sedimenting suspension of weakly flexible fibres. *J. Fluid Mech.*, 756:935–964, 2014.

[23] L. Li and S. E. Spagnolie. Swimming and pumping of rigid helical bodies in viscous fluids. *Phys. Fluids*, 26, 2014.

[24] L. Li and S. E. Spagnolie. Swimming and pumping by helical waves in viscous and viscoelastic fluids. *Phys. Fluids*, 27(2):021902, 2015.

[25] L. Li, X. Xu, and S. E. Spagnolie. A locally gradient-preserving reinitialization for level set functions. *J. Sci. Comput.*, 71(1):274–302, 2017.

[26] L. Li. An Eulerian formulation of immersed interface method for membranes with tangential stretching. *arXiv preprint arXiv:1511.07052*, 2015.

[27] Flavien Léger. A geometric perspective on regularized optimal transport. *arXiv preprint arXiv:1703.10243*, 2017.

[28] Martin Arjovsky, Soumith Chintala, and Léon Bottou. Wasserstein GAN. *arXiv preprint arXiv:1701.07875*, 2017.