1 First prob

Let $f \in H^{1/2}(\mathbb{T}^1)$. Show $e^f \in L^2(\mathbb{T}^1)$

This is one of the problems from the final of Math725 when I took it. It's about the critical case of Soblev embedding.($W^{1/2,2}$)

In the version of proof I used(after discussion with others), the key inequality is Hausdorff-Young inequality.

The known condition is equivalent to

$$\sum |n||\hat{f}_n|^2 < \infty$$

This is critical because if $\hat{f}$ is in L^1, f is in C_0(decaying at infinity) and $\sum |\hat{f}_n| \leq \|\hat{f}n^\alpha\|_{L^p}\|n^{-\alpha}\|_{L^q}$. We see that if $\alpha = 1/2, p = 2$, one has the l^2 norm of above the series but the other series is harmonic which is exactly the critical case. Using the same technique, one can show easily that $\hat{f}$ is in l^q where $1 < q < 2$, which implies that $\hat{f}$ is in all l^q with $q > 1$.

To get a finer estimate, define

$$a_n = (1 + n^2)^{1/4}\hat{f}_n$$

which is in l^2. First of all, letting $1 < q < 2$, one has

$$\sum |\hat{f}_n|^q \leq (\sum |a_n|^2)^{q/2}\left(\sum (1 + n^2)^{-q/(2(2-q))}\right)^{(2-q)/2}$$

Letting $1/q + 1/p = 1$, we have

$$\|\hat{f}\|_{l^q} \leq C(\sum n^{-p/(p-2)})^{(p-2)/(2p)} \leq Dp^{1/2}$$

Housdorff-Young says that

$$\|f\|_{L^p} \leq \|\hat{f}\|_{l^q}$$

We see that

$$\|f^n\|_{L^2}^2 \leq \|f\|_{L^{2n}}^{2n} \leq D^{2n}(2n)^n \Rightarrow \|f^n\|_{L^2} \leq D_1^n n^{n/2}$$

Lastly $\sum_n \frac{D^n n^{n/2}}{n!}$ converges.

The following proof scheme is essentially the same, provided by the instructor:

First of all, for any $q > 1$ $q \sim 1 (q < 2)$, one has

$$\sum |\hat{f}_n|^q \leq (\sum |\hat{f}_n|^2 n)^{q/2}(\sum n^{-q/(2-q)})^{(2-q)/2}$$

implying $\|\hat{f}\|_q \le \|f\|_{H^{1/2}}(\frac{C}{q-1})^{(2-q)/(2q)}$.

At last, use the Young's inequality saying that

$$\|\hat{f} * \hat{g}\|_r \le \|\hat{f}\|_p \|\hat{g}\|_q$$

with $1 + 1/r = 1/p + 1/q$.

Applying this n times, we need $1 + 1/2 = 1/p_n + 1/q_1, 1 + 1/q_1 = 1/p_n + 1/q_2, \cdots, 1 + 1/q_{n-1} = 1/p_n + 1/p_n$. We need $(n-1) + 1/2 = n/p_n$ and thus

$$\|\hat{f} * \hat{f} * \cdots * \hat{f}\|_2 \le \|\hat{f}\|_{p_n}^n$$

resulting in the same estimates.

At last, we see a strange thing: $s * f$ is also in $H^{1/2}$ and thus e^{sf} is in L^2 which implies that e^{2sf} is integrable. Does this mean e^f is in any L^p since e^{pf} is integrable?

2 The second

Let $K(x) = \frac{1}{\sqrt{|x|}(1+x^2)}$ and $K_n(x) = nK(nx), n \ge 1$. Let $f \in L^1(\mathbb{R})$ and define $T_n(f) = K_n * f$. Show that $\sup_n |T_n(f)|(x) < \infty. a.e.$.

One method is to use the maximal function. One shows that

$$|T_n(f)(x)| \le CM(f)(x)$$

where C is independent of n then done, since $M(f)$ is weakly L^1(not L^1) and $M(f)$ is finite almost everywhere.(See Hardy-Littlewood maximal inequality)

Then, one needs to show that such C exists.

In general, assume $\phi(x) = h(|x|)$ and $h \in C^1((0,\infty)), h'(r) < 0$ with $r^d h(r) \to 0$ as $r \to 0, r \to \infty$. One can define

$$\begin{aligned}
A_\epsilon f = \frac{1}{\epsilon^d}\phi(\cdot/\epsilon) * |f| &= \int_0^\infty \frac{r^{d-1}}{\epsilon^d} \int_{S^{n-1}} \phi(r\omega/\epsilon)|f(x-\omega r)| d\omega dr \\
&= -\int_0^\infty \frac{r^{d-1}}{\epsilon^d} \int_{S^{d-1}} |f(x-\omega r)| \int_{r/\epsilon}^\infty h'(t) dt d\omega dr \\
&= -\int_0^\infty \int_0^{\epsilon t} \int_{S^{d-1}} \frac{r^{d-1}}{\epsilon^d} |f(x-\omega r)| h'(t) d\omega dr dt
\end{aligned}$$

Notice that

$$\int_0^{\epsilon t} \frac{r^{d-1}}{\epsilon^d t^d} \int_{S^{d-1}} |f(x-r\omega)| d\omega dr \le M(f)$$

That means

$$C = -\int_0^\infty h'(t)t^d dt < \infty$$

since $\phi \in L^1$