In applications like water flow through a porous medium, one has to solve an elliptic equation $-\nabla \cdot (a(x)\nabla u) = f$ with heterogeneous coefficients $a(x)$ that vary on a characteristic length scale $\ell$ much smaller than the domain size. The coefficients $a(x)$ are typically characterized in statistical terms: Their statistics are assumed to be stationary (i.e. translation invariant) and to decorrelate over distances large compared to $\ell$. In this situation, it is known from the theory of stochastic homogenization that the solution operator, i.e. the inverse of the elliptic operator $-\nabla \cdot a(x)\nabla$, behaves – on scales large compared to $\ell$ – like the inverse of $-\nabla \cdot a_{\text{hom}}\nabla$ with homogeneous, deterministic coefficients $a_{\text{hom}}$. This is a major reduction in complexity.

Hence the relation between the statistics of the heterogeneous coefficients $a(x)$ and the value of the homogenized coefficient $a_{\text{hom}}$ is of practical importance. Stochastic homogenization also provides a formula for $a_{\text{hom}}$ in terms of the ensemble of $a(x)$. It involves the solution of the “corrector problem” $-\nabla \cdot (a(x)(\nabla \phi_\xi + \xi)) = 0$ in the whole space $\mathbb{R}^d$ for a given direction $\xi$. The formula then is given by $a_{\text{hom}}\xi = \langle a(\nabla \phi_\xi + \xi) \rangle$, where $\langle \cdot \rangle$ denotes the ensemble average. Despite its simplicity, this formula has to be approximated in practice:

1) The corrector problem can only be solved for a small number of realizations of the coefficients $a(x)$. Thus, appealing to ergodicity, the ensemble average has to be replaced by a spatial average over a region of large diameter $L$.

2) The corrector problem can only be solved in a finite domain of large diameter $L$, thereby introducing some artificial boundary conditions.

In this talk, we present optimal estimates on both errors for the simplest possible model problem: We consider a discrete elliptic equation on the $d$-dimensional lattice $\mathbb{Z}^d$ with random coefficients $a$, which are identically distributed and independent from edge to edge. This makes a connection with the area of “random walks in random environments” and the area of “gradient fields”. We establish the optimal scaling of both errors in the ratio $L/\ell \gg 1$ (where the correlation length $\ell$ is unity in our model problem). It turns out that the scaling is the same as in the case of coefficients $a(x)$ that are very close to the identity (where the corrector problem can be linearized).
Hence with respect to the error scaling, the highly nonlinear relation between $a(x)$ and $a_{hom}$ behaves like the linearized one.

Our methods involve spectral gap estimates and estimates on the Green’s function which only depend on the ellipticity ratio $\lambda$ of $a$.

This is joint work with Antoine Gloria, INRIA Lille.