A NOTE ON THE STABLE BERNSTEIN THEOREM

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Let $(M^n, \overline{g}) \hookrightarrow \mathbb{R}^{n+1}$ be an immersed, complete, connected, two-sided, stable minimal hypersurface. The Bernstein problem asks whether M is a flat hyperplane.

When M is graphical, the problem was solved for $n \leqslant 7$ in [4, 15, 13, 1, 21], while a counterexample when $n \geqslant 8$ was constructed in [5]. In [5] the authors also construct an immersed, two-sided, stable minimal hypersurfaces $M^7 \hookrightarrow \mathbb{R}^8$ that is not a hyperplane.

The $n = 2$ case of the Bernstein problem has been solved by [8, 14, 19]. Recently, the $n = 3$ case was resolved by different methods in [10, 11, 9]. The idea in [11] has been pushed to $n = 4$ and $n = 5$ in the very recent works [12], and [17], respectively. These proofs show that M has Euclidean volume growth. Hence, leveraging the Schoen–Simon–Yau curvature estimate [20], it follows that M must be flat. When $n = 6$, the curvature estimate is proved in [3] assuming that M has extrinsic Euclidean volume growth.

The aim of this note is to give a new point of view on the recent proofs [11, 12, 17], and to show an obstruction when $n = 6$. Let us denote $\overline{\Delta}, \overline{A}$ the Laplacian, and the second fundamental form of M, respectively. Let u be a positive eigenfunction of the stability operator, so that

$$\overline{\Delta} u \leqslant -|\overline{A}|^2 u. \tag{1}$$

Let r be the distance function from the origin on $\mathbb{R}^{n+1}$. We consider a direct μ-bubble construction on $(M, \overline{g})$, associated to the energy

$$E(\Omega) := \int_{\partial\Omega} r^{-\beta} u^{\gamma} - \int_{\Omega} h r^{-\beta} u^{\gamma}, \tag{2}$$

for some constants $\beta, \gamma > 0$ to be chosen. For a fixed $r_0 > 0$, we will make the choice $h = r^{-1} \cot\big(\varepsilon \log(r/r_0)\big)$,[1] with $\varepsilon \ll 1$. Hence, minimizing (2) gives a stable μ-bubble in the annular region $\{r_0 < r < \pi^{1/\varepsilon} r_0\}$, for every $r_0 > 0$. Denoting $\Sigma := \partial\Omega$, we aim at showing that, for some conformal change $\widetilde{g} = r^{-2p} g$ of Σ (when $n \in \{3, 4\}$ we can take $p = 0$), the μ-bubble stability inequality implies the spectral Ricci lower bound[2]

$$\int_{\Sigma} \left(\frac{n-2}{n-3} |\widetilde{\nabla} \eta^2| + \widetilde{\mathrm{Ric}} \cdot \eta^2 \right) \mathrm{d}\widetilde{V} \geqslant \lambda \int_{\Sigma} \eta^2 \mathrm{d}\widetilde{V}, \qquad \forall\, \eta \in C^{\infty}(\Sigma), \tag{3}$$

for some $\lambda > 0$. Then by [2, Theorem 1], $(\Sigma, \widetilde{g})$ will enjoy a volume upper bound depending on n, λ. Hence, going back to $\overline{g}$, and using the Michael–Simon isoperimetric inequality [18, 6], M shall have Euclidean volume growth. Notice that $\frac{n-2}{n-3}$ is the largest constant that one can put in (3) for this strategy to work, due to [2, Remark 4].

The derivation from the stability inequality of (2) to (3) eventually boils down to showing that three matrices are negative definite for some careful choice of parameters.

Let us point out the following relation between (2) and the μ-bubble arguments in [10, 12, 17]. Suppose u satisfies (1). Then, after the Gulliver-Lawson conformal change, one can choose the spectral biRicci eigenfunction in [12, 17] to be $r^{\frac{n-2}{2}} u$. Taking this

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[1]This works when M is properly immersed. In general one needs a more complicated h, see Lemma 3.

[2]When $n = 3$, the constant $\frac{n-2}{n-3}$ in (3) can be substituted with a sufficiently large number.

transformation, and the conformal change into account, the standard μ-bubble energies in [12, 17] are thus transformed to (2) for some choices of β, γ.

The bi-Ricci curvature, which appeared in [12, 17], does not play a role in our argument. Finally, when $n = 6$, there is no choice of parameters so that the present strategy works; see Lemma 8. This seems to suggest that for $n \geqslant 6$ a new idea has to be introduced if one is willing to show that M has Euclidean volume growth.

Notations. In this note we assume $n \geqslant 3$. We use $\overline{g}$ to denote the metric on the stable minimal hypersurface M, and $\overline{A}, \overline{\nabla}, \overline{\Delta}$, etc, for the objects associated with $(M, \overline{g})$. We use g to denote the metric on the μ-bubble Σ, and A, ∇, Δ, etc, for the objects associated with (Σ, g). The objects after the conformal change on Σ will be denoted $\widetilde{g}, \widetilde{\mathrm{Ric}}, \widetilde{\nabla}$, etc. For a function f, and vector field X, we write $f_X := \partial f / \partial X$. Following [2], we use $\mathrm{Ric}(x)$ to denote the minimal eigenvalue of the Ricci tensor at the point x. We denote with ν the unit normal to the $(n-1)$-dimensional manifold Σ in M, and with $\overline{\nu}$ the unit normal to the n-dimensional manifold M in $\mathbb{R}^{n+1}$. By changing coordinates we can and we shall assume $0 \in M$. We denote with $B_M(p, r)$ the ball (with center $p \in M$ and radius $r > 0$) in the geodesic distance induced by $\overline{g}$ on M.

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1. Preliminaries

Lemma 1 ([12, Proposition 3.2])**.** *Let M^n be a minimal hypersurface in $\mathbb{R}^{n+1}$. Then:*

(1) $\overline{\nabla}^2 r = r^{-1}\overline{g} - r^{-1}\mathrm{d}r^2 - r_{\overline{\nu}}\overline{A}$,

(2) $\overline{\Delta} r = n r^{-1} - r^{-1}|\overline{\nabla} r|^2$.

Lemma 2 ([16, Theorem 7.30])**.** *Let Σ^{n-1} be a manifold and $f \in C^\infty(\Sigma)$. Consider the conformal metric $\widetilde{g} = e^{2f} g$. Then we have*

$$\widetilde{\mathrm{Ric}} = \mathrm{Ric} - (n-3)\nabla^2 f + (n-3)\mathrm{d}f^2 - (\Delta f) g - (n-3)|\nabla f|^2 g. \tag{4}$$

Lemma 3. *Let $0 < \varepsilon < 1$ and $r_0 > 0$ be fixed, and let d denote the distance function from $0 \in M$ computed with respect to the metric $\overline{g}$. Let ρ be a smooth function on $M \setminus \overline{B_M(0, r_0)}$ such that*

$$\rho = r_0 \text{ on } \partial B_M(0, r_0), \qquad (1-\varepsilon)\mathrm{d} \leqslant \rho, \qquad |\overline{\nabla}\rho| \leqslant 1 + \varepsilon. \tag{5}$$

Set $C_\varepsilon := \varepsilon \frac{1-\varepsilon}{1+\varepsilon}$ and define

$$h := r^{-1} \cot\big(C_\varepsilon \log(r_0^{-1}\rho)\big), \tag{6}$$

in the region $\rho^{-1}([r_0, r_0 e^{\pi/C_\varepsilon}]) \subset \overline{B_M\left(0, \frac{1}{1-\varepsilon} r_0 e^{\pi/C_\varepsilon}\right)} \setminus B_M(0, r_0)$. Then, for any unit vector field ν we have

$$-h_\nu - h r^{-1} r_\nu - \varepsilon h^2 - \varepsilon r^{-2} \leqslant 0.$$

Proof. First of all notice that, since M is an isometric immersion in $\mathbb{R}^{n+1}$, we have $\mathrm{d} \geqslant r$. Now, calling $A := C_\varepsilon \log(r_0^{-1}\rho)$ for the ease of notation, we may directly calculate:

$$\begin{aligned} -h_\nu - h r^{-1} r_\nu - \varepsilon h^2 - \varepsilon r^{-2} &= -\big(- r^{-2} r_\nu \cot(A) + r^{-1}\cot'(A) C_\varepsilon \rho_v \rho^{-1}\big) \\ &\quad - r^{-2} r_\nu \cot(A) - r^{-2}\varepsilon \cot^2(A) - \varepsilon r^{-2} \\ &= r^{-2}\Big(\frac{C_\varepsilon \rho_v r \rho^{-1}}{\sin^2(A)} - \varepsilon \frac{\cos^2(A)}{\sin^2(A)} - \varepsilon \Big) \leqslant 0, \end{aligned} \tag{7}$$

where the last inequality is true because $C_\varepsilon \rho_v r \rho^{-1} \leqslant C_\varepsilon |\overline{\nabla}\rho| \mathrm{d}\rho^{-1} \leqslant \varepsilon$, by (5). $\square$

2. The μ-bubble construction

Let $(M^n, \overline{g})$ be a complete immersed two-sided stable minimal surface in $\mathbb{R}^{n+1}$. Thus we have

$$\int_M |\overline{\nabla}\varphi|^2 - |\overline{A}|^2\varphi^2 \geqslant 0, \qquad \forall \varphi \in C_0^\infty(M).$$

Therefore, we can choose a positive $u \in C^\infty(M)$ such that

$$\overline{\Delta} u \leqslant -|\overline{A}|^2 u.$$

For constants $\beta, \gamma > 0$ to be determined, consider the μ-bubble defined by minimizing the energy

$$E(\Omega) = \int_{\partial\Omega} r^{-\beta}u^\gamma - \int_\Omega h r^{-\beta} u^\gamma. \tag{8}$$

Let $\Sigma := \partial\Omega$ be the boundary of the μ-bubble. Let g, A, H, ν denote the induced metric, second fundamental form, mean curvature, and outer unit normal of $\Sigma^{n-1} \hookrightarrow M^n$. Below we make the agreement that all the integrals are taken on Σ with respect to the volume form induced by g.

Given any $\varphi \in C^\infty(\Sigma)$, consider a variation $\{\Sigma_\tau\}$ of Σ, so that the variational vector field at $\tau = 0$ equals $\varphi\nu$. Since Σ is a critical point of (8), the first variation gives

$$0 = \frac{\mathrm{d}E(\Sigma_\tau)}{\mathrm{d}\tau}\Big|_{\tau=0} = \int r^{-\beta}u^\gamma\varphi\Big[H - \beta r^{-1} r_\nu + \gamma u^{-1}u_\nu - h\Big].$$

Hence

$$H = h - \gamma u^{-1}u_\nu + \beta r^{-1} r_\nu. \tag{9}$$

Denote $Y := u^{-1}u_\nu$, $Z := r^{-1}r_\nu$, so we have $H = h - \gamma Y + \beta Z$.

2.1. **Source terms from the stability inequality.** By calculating the second variation $0 \leqslant \frac{\mathrm{d}^2 E}{\mathrm{d}\tau^2}|_{\tau=0}$, we obtain the following:

$$\begin{aligned}0 \leqslant \int r^{-\beta}u^\gamma\varphi\Big[&-\Delta\varphi - |A|^2\varphi - \overline{\mathrm{Ric}}_{\nu\nu}\varphi + \beta r^{-2}r_\nu^2\varphi - \beta r^{-1}\varphi\overline{\nabla}^2 r(\nu,\nu) + \beta r^{-1}\langle\nabla r, \nabla\varphi\rangle\\ &- \gamma u^{-2}u_\nu^2\varphi + \gamma u^{-1}\varphi\overline{\Delta}u - \gamma u^{-1}\varphi\Delta u - \gamma u^{-1}\varphi H u_\nu - \gamma u^{-1}\langle\nabla u, \nabla\varphi\rangle - h_\nu\varphi\Big].\end{aligned}$$

There are two ways to expand the term $\overline{\nabla}^2 r(\nu,\nu)$, thus we set a new variable to keep this degree of freedom. For a constant $t \in \mathbb{R}$ to be determined, we split

$$\begin{aligned}\overline{\nabla}^2 r(\nu,\nu) &= t\big(\overline{\Delta} r - \Delta r - H r_\nu\big) + (1-t)\overline{\nabla}^2 r(\nu,\nu)\\ &= ntr^{-1} - tr^{-1}|\overline{\nabla} r|^2 - t\Delta r - tHr_\nu + (1-t)r^{-1}\\ &\qquad - (1-t)r^{-1}r_\nu^2 - (1-t)r_{\overline{\nu}}\overline{A}_{\nu\nu}.\end{aligned} \tag{10}$$

Plugging this into the stability inequality, we obtain:

$$\begin{aligned}0 \leqslant \int r^{-\beta}u^\gamma\varphi\Big[&-\Delta\varphi - |A|^2\varphi - \overline{\mathrm{Ric}}_{\nu\nu}\varphi + \beta r^{-2}r_\nu^2\varphi\\ &+ \big[-(nt - t + 1)\beta r^{-2}\varphi + t\beta r^{-2}|\overline{\nabla} r|^2\varphi\big] + t\beta r^{-1}\varphi\Delta r + t\beta r^{-1}\varphi H r_\nu\\ &+ (1-t)\beta r^{-2}r_\nu^2\varphi + (1-t)\beta r^{-1}r_{\overline{\nu}}\overline{A}_{\nu\nu}\varphi + \beta r^{-1}\langle\nabla r, \nabla\varphi\rangle\\ &- \gamma u^{-2}u_\nu^2\varphi + \gamma u^{-1}\varphi\overline{\Delta}u - \gamma u^{-1}\varphi\Delta u - \gamma u^{-1}\varphi H u_\nu - \gamma u^{-1}\langle\nabla u, \nabla\varphi\rangle - h_\nu\varphi\Big].\end{aligned}$$

There are sixteen terms in this integral, let us denote them by [[1]] $\sim$ [[16]] respectively. For another test function $\psi \in C^\infty(\Sigma)$, set $\varphi := r^{\beta/2}u^{-\gamma/2}\psi$. We simplify each term as follows: (for better readability, we will drop the integration sign in all the computations below; the symbol $\overset{*}{=}$ means we are using integration by parts)

$$
\begin{aligned}
[[1]] &= -r^{-\beta/2}u^{\gamma/2}\psi\Delta(r^{\beta/2}u^{-\gamma/2}\psi) \overset{*}{=} \nabla(r^{-\beta/2}u^{\gamma/2}\psi)\cdot\nabla(r^{\beta/2}u^{-\gamma/2}\psi),\\
[[2]] &= -|A|^2\psi^2,\\
[[3]] &= -\overline{\mathrm{Ric}}_{\nu\nu}\psi^2 = \overline{A}^2(\nu,\nu)\psi^2 \text{ (where } \overline{A}^2_{ij} := \overline{A}_{ip}\overline{g}^{pq}\overline{A}_{qj}),\\
[[4]] &= \beta Z^2\psi^2,\\
[[5]] &= -(nt-t+1)\beta r^{-2}\psi^2 + t\beta r^{-2}\psi^2|\overline{\nabla}r|^2,\\
[[6]] &= t\beta r^{-1}\psi^2\Delta r \overset{*}{=} -t\beta\nabla(r^{-1}\psi^2)\cdot\nabla r,\\
[[7]] &= t\beta\psi^2(h-\gamma Y+\beta Z)Z,\\
[[8]] &= (1-t)\beta\psi^2 Z^2,\\
[[9]] &= (1-t)\beta\psi^2 r^{-1} r_{\overline{\nu}}\overline{A}_{\nu\nu},\\
[[10]] &= \beta r^{-\beta/2}u^{\gamma/2}\psi r^{-1}\nabla r\cdot\nabla(r^{\beta/2}u^{-\gamma/2}\psi),\\
[[11]] &= -\gamma Y^2\psi^2,\\
[[12]] &= \gamma u^{-1}\psi^2\overline{\Delta}u \leqslant -\gamma|\overline{A}|^2\psi^2,\\
[[13]] &= -\gamma u^{-1}\psi^2\Delta u \overset{*}{=} \gamma\nabla(u^{-1}\psi^2)\cdot\nabla u,\\
[[14]] &= -\gamma\psi^2(h-\gamma Y+\beta Z)Y,\\
[[15]] &= -\gamma r^{-\beta/2}u^{\gamma/2}\psi u^{-1}\nabla u\cdot\nabla(r^{\beta/2}u^{-\gamma/2}\psi),\\
[[16]] &= -h_\nu\psi^2.
\end{aligned}
$$

2.2. **Target terms from spectral Ricci curvature.** For $p\in\mathbb{R}$, consider the conformal metric $\widetilde{g} = r^{-2p}g$ on the μ bubble Σ. Let $\widetilde{e}$ be a $\widetilde{g}$-unit measurable vector field such that $\widetilde{\mathrm{Ric}}(\widetilde{e},\widetilde{e}) = \widetilde{\mathrm{Ric}}$ everywhere on Σ. Then set $e = r^{-p}\widetilde{e}$ (which is g-unit). Let $\{e_i\}_{2\leqslant i\leqslant n-1}$ be measurable g-unit vector fields that form an orthonormal frame on Σ along with e.

We aim at showing that the stability inequality implies the spectral Ricci bound

$$
s\int\Big(\frac{n-2}{n-3}|\widetilde{\nabla}\eta|^2 + \widetilde{\mathrm{Ric}}\cdot\eta^2\Big)\,\mathrm{d}\widetilde{V} \geqslant \varepsilon\int\eta^2\,\mathrm{d}\widetilde{V}, \tag{11}
$$

on Σ, for some choice $s>0$[3] and sufficiently small $\varepsilon>0$. When $n=3$ one may replace $\frac{n-2}{n-3}$ with a sufficiently large positive number, and the same strategy still works.

The first step for achieving our goal is to convert (11) back to the original metric g, and obtain a target expression. The integrals below are on Σ, and when we omit the volume form, we are assuming the integration is with respect to g. Using Lemma 2 with $f=-p\log r$, we have

$$
\begin{aligned}
J &:= \int\Big(\frac{n-2}{n-3}s|\widetilde{\nabla}\eta|^2 + s\widetilde{\mathrm{Ric}}(\widetilde{e},\widetilde{e})\eta^2\Big)\,\mathrm{d}\widetilde{V} = \int r^{(3-n)p}\Big(\frac{n-2}{n-3}s|\nabla\eta|^2 + s\widetilde{\mathrm{Ric}}(e,e)\eta^2\Big)\\
&= \int\frac{n-2}{n-3}sr^{(3-n)p}|\nabla\eta|^2 + r^{(3-n)p}s\Big[\mathrm{Ric}(e,e) + (n-3)p\nabla^2\log r(e,e)\\
&\qquad + (n-3)p^2|\nabla_e\log r|^2 + p\Delta\log r - (n-3)p^2|\nabla\log r|^2\Big]\eta^2.
\end{aligned}
$$

To remove the powers on r, we make the choice $\eta := r^{\frac{n-3}{2}p}\psi$ and simplify:

$$
\begin{aligned}
J = \int&\frac{n-2}{n-3}s|\nabla\psi|^2 + (n-2)sp\psi\nabla\psi\cdot\nabla\log r + \frac{1}{4}(n-2)(n-3)sp^2\psi^2|\nabla\log r|^2\\
&+ s\psi^2\,\mathrm{Ric}(e,e) + (n-3)sp\psi^2\nabla^2\log r(e,e) + (n-3)sp^2\psi^2r^{-2}r_e^2\\
&- 2sp\psi\nabla\psi\cdot\nabla\log r - (n-3)sp^2\psi^2|\nabla\log r|^2.
\end{aligned}
$$

[3]The appearance of α-biRicci curvature in [17] corresponds to the matching parameter s in (11).

There are two complex terms. We rewrite the first using Gauss' equations:

$$
\begin{aligned}
s\,\mathrm{Ric}(e,e) &= s\sum_i R_{eiie} = s\sum_i \left(\overline{R}_{eiie} + A_{ee}A_{ii} - A_{ei}^2\right)\\
&= s\sum_i \left(\overline{A}_{ee}\overline{A}_{ii} - \overline{A}_{ei}^2\right) + s\sum_i \left(A_{ee}A_{ii} - A_{ei}^2\right),
\end{aligned}
$$

and for the second we notice:

$$
\begin{aligned}
(n-3)sp\psi^2\nabla^2\log r(e,e) &= (n-3)sp\psi^2\Big[\overline{\nabla}^2\log r(e,e) - A_{ee}r^{-1}r_\nu\Big]\\
&= (n-3)sp\psi^2\Big[r^{-2} - 2r^{-2}r_e^2 - r^{-1}r_{\overline{\nu}}\overline{A}_{ee} - r^{-1}r_\nu A_{ee}\Big].
\end{aligned}
$$

Combining the expressions of the same type, we are led to define the following terms $[[17]] \sim [[25]]$, that satisfy $\sum_{k=17}^{25}[[k]] + s\int\left(\frac{n-2}{n-3}|\widetilde{\nabla}\eta|^2 + \widetilde{\mathrm{Ric}}(\widetilde{e},\widetilde{e})\eta^2\right)\mathrm{d}\widetilde{V} = 0$. As usual, we omit the integral sign in the expressions.

$[[17]] = -\frac{n-2}{n-3}s|\nabla\psi|^2$,

$[[18]] = -(n-4)sp\psi\nabla\psi\cdot\nabla\log r$,[4]

$[[19]] = s\psi^2\sum_i\left(\overline{A}_{ei}^2 - \overline{A}_{ee}\overline{A}_{ii}\right)$,

$[[20]] = s\psi^2\sum_i\left(A_{ei}^2 - A_{ee}A_{ii}\right)$,

$[[21]] = (n-3)sp\psi^2r^{-1}r_{\overline{\nu}}\overline{A}_{ee}$,

$[[22]] = (n-3)sp\psi^2r^{-1}r_\nu A_{ee}$,

$[[23]] = \frac{1}{4}(6-n)(n-3)sp^2\psi^2r^{-2}|\nabla r|^2$,

$[[24]] = (n-3)s(2p-p^2)\psi^2r^{-2}r_e^2$,

$[[25]] = -(n-3)sp\psi^2r^{-2}$.

2.3. **Algebraic reductions.** We aim to arrange the terms $[[1]] \sim [[25]]$ into three groups, and turn the proof of the main theorem into the negative definiteness of three corresponding matrices. As usual, we suppress the integral signs in this subsection.

We start with collecting the miscellaneous terms:

$$
I_R = [[5]] + [[16]] + [[23]] + [[24]] + [[25]]. \tag{12}
$$

Let ε denote a sufficiently small number. Notice that $|\nabla r|^2 + r_\nu^2 + r_{\overline{\nu}}^2 = 1$. We use this to decompose $[[5]]$ and $[[25]]$, and obtain (recall $Z = r^{-1}r_\nu$)

$$
\begin{aligned}
\psi^{-2}I_R \leqslant{}& \Big[-h_\nu - hr^{-1}r_\nu - \varepsilon h^2 - 2\varepsilon r^{-2}\Big] + C_1r^{-2}|\nabla r|^2 + C_2r^{-2}r_{\overline{\nu}}^2 + C_3Z^2 + hZ\\
&+\Big[2\varepsilon r^{-2}|\nabla r|^2 + 2\varepsilon r^{-2}r_{\overline{\nu}}^2 + \varepsilon h^2 + 2\varepsilon Z^2\Big],
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
C_1 &:= -\big[nt-2t+1\big]\beta - (n-3)sp + (n-3)s\max\left\{2p-p^2,0\right\} &(14)\\
&\quad + \frac{1}{4}(6-n)(n-3)sp^2, &(15)\\
C_2 &:= -\big[nt-t+1\big]\beta - (n-3)sp, &(16)\\
C_3 &:= -\big[nt-2t+1\big]\beta - (n-3)sp. &(17)
\end{aligned}
$$

The reason to add a $-hr^{-1}r_\nu$ term in (13) is clear by Lemma 3.

[4]When $n=3$, this term is $2sp\psi\nabla\psi\cdot\nabla\log r$, thus the $(1,3)$-entry of (18) is $-\frac{1}{2}\beta(2t-1)+sp$.

Lemma 4 (Gradient terms)**.** *Set the matrix*

$$P_\nabla = \begin{pmatrix} 1-\frac{n-2}{n-3}s & \frac{\gamma}{2} & -\frac{\beta(2t-1)+(n-4)sp}{2} \\ \frac{\gamma}{2} & \frac{\gamma^2}{4}-\gamma & -\frac{\beta\gamma}{4} \\ -\frac{\beta(2t-1)+(n-4)sp}{2} & -\frac{\beta\gamma}{4} & \frac{\beta^2}{4}+t\beta+C_1 \end{pmatrix}, \tag{18}$$

and, for $\varepsilon > 0$, collect the gradient terms

$$I_\nabla := [[1]] + [[6]] + [[10]] + [[13]] + [[15]] + [[17]] + [[18]] + (C_1+2\varepsilon)\psi^2 r^{-2}|\nabla r|^2. \tag{19}$$

If $P_\nabla < 0$, then for sufficiently small $\varepsilon > 0$ we have $I_\nabla \leqslant 0$.

Proof. Set $\theta = r^{-\beta/2}u^{\gamma/2}$. We compute (as usual, we suppress the integral sign for the simplicity of expressions):

$$[[1]] = \nabla(\theta\psi)\cdot\nabla(\theta^{-1}\psi) = |\nabla\psi|^2 - \psi^2|\nabla\log\theta|^2, \tag{20}$$

and

$$\begin{aligned} [[6]] &= -2t\beta\psi\nabla\psi\cdot\nabla\log r + t\beta\psi^2|\nabla\log r|^2, \\ [[13]] &= 2\gamma\psi\nabla\psi\cdot\nabla\log u - \gamma\psi^2|\nabla\log u|^2 \end{aligned} \tag{21}$$

and

$$\begin{aligned} [[10]]+[[15]] &= \beta\theta\psi\nabla\log r\cdot\nabla(\theta^{-1}\psi) - \gamma\theta\psi\nabla\log u\cdot\nabla(\theta^{-1}\psi) \\ &= \beta\psi\nabla\psi\cdot\nabla\log r - \gamma\psi\nabla\psi\cdot\nabla\log u + 2\psi^2|\nabla\log\theta|^2. \end{aligned} \tag{22}$$

We also expand

$$\psi^2|\nabla\log\theta|^2 = \frac{\beta^2}{4}\psi^2|\nabla\log r|^2 + \frac{\gamma^2}{4}\psi^2|\nabla\log u|^2 - \frac{\beta\gamma}{2}\psi^2\nabla\log r\cdot\nabla\log u. \tag{23}$$

Assembling (20) $\sim$ (23) and adding up with $[[17]]+[[18]]+(C_1+2\varepsilon)\psi^2|\nabla\log r|^2$, we get[5]

$$\begin{aligned} \psi^{-2}I_\nabla = &\Big(1-\frac{n-2}{n-3}s\Big)\frac{|\nabla\psi|^2}{\psi^2} + \big(\frac{\gamma^2}{4}-\gamma\big)|\nabla\log u|^2 + \big(\frac{\beta^2}{4}+t\beta+C_1+2\varepsilon\big)|\nabla\log r|^2 \\ &+\gamma\frac{\nabla\psi}{\psi}\cdot\nabla\log u - \big[\beta(2t-1)+(n-4)sp\big]\frac{\nabla\psi}{\psi}\cdot\nabla\log r - \frac{\beta\gamma}{2}\nabla\log u\cdot\nabla\log r. \end{aligned}$$

The matrix P_∇ is exactly the coefficient matrix of this quadratic form subtracted with $\mathrm{diag}(0,0,2\varepsilon)$. Thus the assertion follows. □

Lemma 5 (Terms involving $\overline{A}$)**.** *Set the matrix*

$$P_{\overline{A}} := \begin{pmatrix} s-\frac{n-1}{n-2}\gamma & \frac{1}{2}\big(s-\frac{2\gamma}{n-2}\big) & \frac{1}{2}(n-3)sp \\ \frac{1}{2}\big(s-\frac{2\gamma}{n-2}\big) & 1-\frac{n-1}{n-2}\gamma & \frac{1}{2}(1-t)\beta \\ \frac{1}{2}(n-3)sp & \frac{1}{2}(1-t)\beta & C_2 \end{pmatrix}. \tag{24}$$

If $2\gamma \geqslant \max\{1,s\}$, and $P_{\overline{A}} < 0$, then for sufficiently small $\varepsilon > 0$ we have

$$I_{\overline{A}} := [[3]] + [[9]] + [[12]] + [[19]] + [[21]] + (C_2+2\varepsilon)\psi^2 r^{-2} r_\nu^2 \leqslant 0. \tag{25}$$

[5] Multiplying ψ^{-2} is only for the ease of expressions, one does not need to assume $\psi \neq 0$.

Proof. If $2\gamma \geqslant \max\{1, s\}$, then we can discard the cross curvature terms and estimate

$$\begin{aligned}
\psi^{-2} I_{\overline{A}} &= \Big[\overline{A}_{\nu\nu}^2 + \overline{A}_{\nu e}^2 + \sum_i \overline{A}_{\nu i}^2\Big] + (1-t)\beta r^{-1} r_{\overline{\nu}} \overline{A}_{\nu\nu} + (n-3) s p r^{-1} r_{\overline{\nu}} \overline{A}_{ee} \\
&\quad + \Big[-\gamma \overline{A}_{\nu\nu}^2 - \gamma \overline{A}_{ee}^2 - \gamma \sum_i \overline{A}_{ii}^2 - 2\gamma \overline{A}_{\nu e}^2 - 2\gamma \sum_i \overline{A}_{\nu i}^2 - 2\gamma \sum_i \overline{A}_{ei}^2 \\
&\quad - 2\gamma \sum_{i<j} \overline{A}_{ij}^2\Big] + \Big[-s\overline{A}_{ee} \sum_i \overline{A}_{ii} + s \sum_i \overline{A}_{ei}^2\Big] + (C_2 + 2\varepsilon) r^{-2} r_{\overline{\nu}}^2 \\
&\leqslant (1-\gamma)\overline{A}_{\nu\nu}^2 - \gamma \overline{A}_{ee}^2 - \gamma \sum_i \overline{A}_{ii}^2 - s\overline{A}_{ee} \sum_i \overline{A}_{ii} \\
&\quad + (1-t)\beta r^{-1} r_{\overline{\nu}} \overline{A}_{\nu\nu} + (n-3) s p r^{-1} r_{\overline{\nu}} \overline{A}_{ee} + (C_2 + 2\varepsilon) r^{-2} r_{\overline{\nu}}^2.
\end{aligned}$$

Then we use the trace inequality and the fact $\sum_i \overline{A}_{ii} + \overline{A}_{ee} + \overline{A}_{\nu\nu} = 0$ to eliminate the $\overline{A}_{ii}$ term. Calling $U = \overline{A}_{ee}$, $V = \overline{A}_{\nu\nu}$, $W = r^{-1} r_{\overline{\nu}}$, this gives the result:

$$\begin{aligned}
\psi^{-2} I_{\overline{A}} \leqslant -\gamma U^2 + (1-\gamma) V^2 - \frac{\gamma}{n-2}(U+V)^2 + sU(U+V) \\
+ (1-t)\beta V W + (n-3) s p U W + (C_2 + 2\varepsilon) W^2.
\end{aligned}$$

The matrix $P_{\overline{A}}$ is exactly the coefficient matrix of the previous quadratic form subtracted with $\operatorname{diag}(0,0,2\varepsilon)$. Thus the assertion follows. □

Lemma 6 (Terms involving A and Y, Z)**.** *Set the matrix*

$$P_{AYZ} := \begin{pmatrix} s - \frac{n-1}{n-2} & \frac{1}{n-2} - \frac{s}{2} & \left(\frac{s}{2} - \frac{1}{n-2}\right)\gamma & a_1 \\ \frac{1}{n-2} - \frac{s}{2} & -\frac{1}{n-2} & -\frac{(n-4)\gamma}{2(n-2)} & a_2 \\ \left(\frac{s}{2} - \frac{1}{n-2}\right)\gamma & -\frac{(n-4)\gamma}{2(n-2)} & \frac{n-3}{n-2}\gamma^2 - \gamma & a_3 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}, \tag{26}$$

where we denote for convenience

$$\begin{aligned}
a_1 &:= \frac{\beta}{n-2} - \frac{s\beta}{2} + \frac{n-3}{2} s p, \qquad & a_2 &:= \frac{1}{2} + \frac{t\beta}{2} - \frac{\beta}{n-2}, \\
a_3 &:= \Big(\frac{1}{n-2} - \frac{t+1}{2}\Big)\beta\gamma, \qquad & a_4 &:= C_3 - \frac{\beta^2}{n-2} + (2-t)\beta + t\beta^2.
\end{aligned}$$

For $\varepsilon > 0$, collect the terms

$$I_A := [[2]] + [[20]] + [[22]], \tag{27}$$

$$I_{YZ} := [[4]] + [[7]] + [[8]] + [[11]] + [[14]] + \psi^2 (C_3 Z^2 + hZ + \varepsilon h^2 + 2\varepsilon Z^2). \tag{28}$$

If $s \leqslant 2$ and $P_{AYZ} < 0$, then for some sufficiently small $\varepsilon > 0$ we have $I_A + I_{YZ} \leqslant 0$.

Proof. Since $s \leqslant 2$, we can drop the cross curvature terms A_{ei}, A_{ij} and get

$$\psi^{-2} I_A \leqslant -A_{ee}^2 - \sum_i A_{ii}^2 - sA_{ee} \sum_i A_{ii} + (n-3) s p A_{ee} Z.$$

Using the trace inequality and noting that $\sum_i A_{ii} = H - A_{ee} = h - \gamma Y + \beta Z - A_{ee}$, we further estimate

$$\begin{aligned}
\psi^{-2} I_A \leqslant -A_{ee}^2 - \frac{1}{n-2}\big(h - \gamma Y + \beta Z - A_{ee}\big)^2 \\
- sA_{ee}\big(h - \gamma Y + \beta Z - A_{ee}\big) + (n-3) s p A_{ee} Z.
\end{aligned} \tag{29}$$

Next, we have

$$\begin{aligned}\psi^{-2}I_{YZ} &= \beta Z^2 + t\beta(h-\gamma Y+\beta Z)Z + (1-t)\beta Z^2 - \gamma Y^2\\ &\qquad - \gamma(h-\gamma Y+\beta Z)Y + C_3 Z^2 + hZ + \varepsilon(h^2+2Z^2)\\ &= (2-t)\beta Z^2 - \gamma Y^2 + (-\gamma Y + t\beta Z)(h-\gamma Y+\beta Z)\\ &\qquad + C_3 Z^2 + hZ + \varepsilon(h^2+2Z^2).\end{aligned}$$

The lemma follows by computing the coefficient matrix of $\psi^{-2}(I_A+I_{YZ})$, with variables $\{A_{ee}, h, Y, Z\}$. □

3. Proof of the Euclidean volume growth and stable Bernstein Theorem

From now on let us make the following assumptions.

Assumption 1. *Let $n \geqslant 3$, and let $\gamma, s, p, \beta > 0$, $t \in \mathbb{R}$ be such that*

$$0 < s \leqslant 2, \quad 2\gamma \geqslant \max\{1, s\}, \quad P_\nabla < 0, \quad P_{\overline{A}} < 0, \quad P_{AYZ} < 0. \tag{30}$$

Notice that if all the conditions in Assumption 1 are met, and we are in the assumptions of Lemma 3, then using Lemma 4 ∼ 6, Lemma 3, and (13), for some $\varepsilon \ll 1$ depending only on $n, \gamma, s, p, \beta, t$, we have

$$0 \leqslant \sum_{k=1}^{16}[[k]] = s\int\Big(\frac{n-2}{n-3}|\widetilde{\nabla}\eta|^2 + \widetilde{\mathrm{Ric}}\cdot\eta^2\Big)\,\mathrm{d}\widetilde{V} + \sum_{k=1}^{25}[[k]] \tag{31}$$

$$\begin{aligned}&\leqslant s\int\Big(\frac{n-2}{n-3}|\widetilde{\nabla}\eta|^2 + \widetilde{\mathrm{Ric}}\cdot\eta^2\Big)\,\mathrm{d}\widetilde{V} + \Big[I_\nabla + I_{\overline{A}} + I_A + I_{YZ}\Big]\\ &\qquad + \int\Big[-h_\nu - hr^{-1}r_\nu - \varepsilon h^2 - 2\varepsilon r^{-2}\Big]\psi^2\mathrm{d}V \end{aligned}\tag{32}$$

$$\leqslant s\int\Big(\frac{n-2}{n-3}|\widetilde{\nabla}\eta|^2 + \widetilde{\mathrm{Ric}}\cdot\eta^2\Big)\mathrm{d}\widetilde{V} - \varepsilon\int r^{-2}\psi^2\,\mathrm{d}V. \tag{33}$$

Lemma 7. *Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be an immersed, complete, connected, simply connected, stable minimal hypersurface with $n \geqslant 3$. Let $\gamma, s, p, \beta > 0$ and $t \in \mathbb{R}$ be such that Assumption 1 is verified. Fix $\varepsilon \ll 1$ so that (32) holds. For each $r_0 > 0$, choose*

$$h = r^{-1}\cot\big(\varepsilon\log(\rho/r_0)\big), \tag{34}$$

where ρ is a smooth function on M satisfying (5). Then there exists a bounded stable μ-bubble $\Omega \supset B_M(0, r_0)$ for the energy (8), such that

$$\mathrm{vol}(\partial\Omega) \leqslant C(s,\varepsilon)r_0^{n-1}.$$

Proof. First of all notice that, given $0 < \varepsilon < 1$, a smooth function ρ satisfying (5) exists by mollifying the Riemannian distance d on $(M, \overline{g})$. By Lemma 3 we have that h is defined on a subset of $\overline{B_M\left(0, \frac{1}{1-\varepsilon}r_0 e^{\pi/C_\varepsilon}\right)} \setminus B_M(0, r_0)$. Moreover, given the explicit expression of h, the standard existence theory for μ-bubbles [22] gives the existence of a bounded stable μ-bubble $\Omega \supset B_M(0, r_0)$ for the energy (8). By taking the connected component of Ω which contains $B_M(0, r_0)$, and by adding all the bounded components of $M \setminus \Omega$ to Ω, we may assume that Ω is connected, and $M \setminus \Omega$ only has unbounded connected components. By [7] we have that M has one end. Then $M \setminus \Omega$ is connected as well.

Since M is simply connected and $\Omega, M \setminus \Omega$ are connected we infer that $\partial\Omega$ is connected. Moreover, using also Lemma 3, by our derivations above, the conformal metric $\widetilde{g} = r^{-2p}g$

on $\partial\Omega$ satisfies the stability inequality (33). Recall the relation $\psi = r^{\frac{3-n}{2}p}\eta$. Taking this transformation and the conformal change into account, (33) is equivalent to

$$\int \big(\frac{n-2}{n-3}|\widetilde{\nabla}\eta|^2 + \widetilde{\mathrm{Ric}} \cdot \eta^2\big)\, \mathrm{d}\widetilde{V} \geqslant s^{-1}\varepsilon \int r^{2p-2}\eta^2\, \mathrm{d}\widetilde{V} \geqslant Cr_0^{2p-2} \int \eta^2\, \mathrm{d}\widetilde{V}.$$

As a result, by [2, Theorem 1], we have the volume bound

$$C' r_0^{-(n-1)(p-1)} \geqslant \mathrm{vol}(\partial\Omega, \widetilde{g}) = \int r^{-(n-1)p}\, \mathrm{d}V \geqslant Cr_0^{-(n-1)p}\, \mathrm{vol}(\partial\Omega).$$

This implies $\mathrm{vol}(\partial\Omega) \leqslant Cr_0^{n-1}$. □

Theorem 1. *Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be an immersed, complete, connected, simply connected, stable minimal hypersurface with $n \geqslant 3$. Assume there are $\gamma, s, p, \beta > 0$, and $t \in \mathbb{R}$ such that Assumption 1 is verified. Then M has Euclidean volume growth.*

Proof. This is a direct consequence of Lemma 7 and Michael–Simon inequality [18] (see also [6]). □

Lemma 8. *If $3 \leqslant n \leqslant 5$, there exist constants satisfying Assumption 1. If $n = 6$, there is no choice of constants for which Assumption 1 is true.*

Proof. By direct verification when $n = 5$ a choice that works is:

$$(s, \gamma, p, \beta, t) = \big(\frac{9}{10}, \frac{9}{10}, \frac{1}{2}, 1, \frac{1}{3}\big).$$

When $n \in \{3, 4\}$, a choice that works is:

$$(s, \gamma, p, \beta, t) = (1, 1, 0, 1, 1).$$

Now suppose $n = 6$. We make the following observations:

(i) The $(1,1)$-entry of P_∇ and P_{AYZ} are negative. These imply $\frac{3}{4} < s < \frac{5}{4}$. The upper left (2×2) block of P_{AYZ} has positive determinant, which implies $s < 1$.

(ii) The $(2,2)$-entry of $P_{\overline{A}}$ and the $(3,3)$-entry of P_{AYZ} are negative. These imply $\frac{4}{5} < \gamma < \frac{4}{3}$.

(iii) The upper left (2×2) block of P_∇ has positive determinant, which implies $s > \frac{3}{4-\gamma}$. Combined with $s < 1$, this implies $\gamma < 1$.

(iv) The upper left (2×2) block of $P_{\overline{A}}$ has positive determinant, which implies

$$6\gamma^2 - 5\gamma > s^2 - 4s + 4\gamma s. \tag{35}$$

The right side of (35) is increasing when $s > 2(1-\gamma)$, therefore combined with $s > \frac{3}{4-\gamma}$ we have

$$6\gamma^2 - 5\gamma > \frac{9}{(4-\gamma)^2} + \frac{12(\gamma-1)}{4-\gamma}.$$

However, this is not possible when $\frac{4}{5} < \gamma < 1$. □

Theorem 2. *Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be an immersed, connected, complete, two-sided, stable minimal hypersurface with $3 \leqslant n \leqslant 5$. Then M is a flat hyperplane.*

Proof. Passing to the universal cover $\widetilde{M}$, it follows from Lemma 8, and Theorem 1 that $\widetilde{M}$, and thus M, has intrinsic Euclidean volume growth. Then Schoen–Simon–Yau's result in [20] concludes the proof. □

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