*Quantum mechanics, Schrödinger operators and spectral theory.* Spectral theory of Schrödinger operators has been my original field of expertise. It is a wonderful mix of functional analysis, PDE and Fourier analysis. In quantum mechanics, every physical observable is described by a self-adjoint operator - an infinite dimensional symmetric matrix. For example, $-\Delta$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$ when defined on an appropriate domain (Sobolev space $H^2$). In quantum mechanics it corresponds to a free particle - just traveling in space. What does it mean, exactly? Well, in quantum mechanics, position of a particle is described by a wave function, $\phi(x,t) \in L^2(\mathbb{R}^d)$. The physical meaning of the wave function is that the probability to find it in a region $\Omega$ at time $t$ is equal to $\int_\Omega |\phi(x,t)|^2\,dx$. You can't know for sure where the particle is for sure. If initially the wave function is given by $\phi(x,0) = \phi_0(x)$, the Schrödinger equation says that its evolution is given by $\phi(x,t) = e^{-i\Delta t}\phi_0$. But how to compute what is $e^{-i\Delta t}$? This is where Fourier transform comes in handy. Differentiation becomes multiplication on the Fourier transform side, and so

$$e^{-i\Delta t}\phi_0(x) = \int_{\mathbb{R}^d} e^{ikx - i|k|^2 t}\hat{\phi}_0(k)\,dk,$$

where $\hat{\phi}_0(k) = \int_{\mathbb{R}^d} e^{-ikx}\phi_0(x)\,dx$ is the Fourier transform of $\phi_0$. I omitted some $\pi$'s since they do not change the picture. Stationary phase methods lead to very precise estimates on free Schrödinger evolution. In particular, for large time,

$$e^{-i\Delta t}\phi_0(x) \sim \frac{1}{t^{d/2}}\hat{\phi}_0\left(\frac{x}{t}\right) + o(t^{-d/2}).$$

Thus one should think of $\hat{\phi}_0(k)$ as of initial distribution of the particle's momentum: the particle travels to infinity at the speeds corresponding to support of $\hat{\phi}_0(k)$ and also disperses in space with time.

But free particle is not very interesting. A much more interesting problem is when we add electric potential, some sort of effective interaction with other particles. The operator describing this system is now $H_V = -\Delta + V(x)$. This is much more interesting. How to analyze this operator? That depends on the structure of $V$. In general, spectral theorem guarantees that if $A$ is a self-adjoint operator on some Hilbert space $H$, than it is unitarily equivalent to a direct sum of operators of multiplication acting on $L^2(\mathbb{R}, d\mu_i)$ with some measures $\mu_i$. For example, for $-\Delta$ Fourier transform $\Phi$ plays a role of such unitary operator: $-\Delta f(x) = \Phi^{-1}|k|^2\Phi f(x)$. So $-\Delta$ is unitarily equivalent to the operator of multiplication by $|k|^2$ on $L^2(dx, \mathbb{R}^d)$ (this is not quite what spectral theorem guarantees in general but is close). Explicit operators of multiplication are among the easiest representations of operators, and all self-adjoint operators can be reduced to this form. The task then becomes studying $\mu$, the spectral measure, which corresponds to the operator, and the transformation $U$ which negotiates between the operator and its easiest representation.

Let me discuss now the simplest possible case: one dimensional Schrödinger operator with decaying potential. It describes a charged quantum particle in a some local electric field. What new phenomena are possible here compared to the free particle case? First of all, there can be bound states - eigenvalues of $H_V$. The particles with energy corresponding to these eigenvalues do not travel to infinity but remain localized near support of the potential. The corresponding

$L^2$ eigenfunctions give probability density for the location of the particle. These eigenvalues can appear only if potential is attractive - negative - in some region. The eigenvalues in this case are negative as well. This is how atoms work: electron energy levels are just eigenvalues of appropriate operators. If the potential is decaying sufficiently fast, particles with positive energies still travel without bound, and are now called scattered states. While the particle is far away from the center of potential, it looks very much as a free wave. As it approaches the potential, part of the particle (I know it sounds weird, but welcome to quantum mechanics) gets reflected, while another part can penetrate the barrier and continue it motion. After a while both parts move away from the support of the potential and look like a free particles again. There is a certain probability that the particle got reflected and a certain probability it got through - but you can't know without making further measurements. This phenomenon is called scattering, and it is related to the fact that if $V(x)$ is, say, zero outside some range, than there are still solutions $\psi_\pm(x,k)$ of the equation $-\psi'' + V(x)\psi = k^2\psi$ which look like Fourier transform. They are called "plane wave" solutions and they satisfy $\psi_\pm(x,k) = \exp(\pm ikx)$ as $x \to +\infty$. One can use these solutions to build an eigenfunction transform akin to the Fourier transform and build a unitary operator that turns $H_V$ into multiplication operator on $L^2$ with Lebesgue measure. Thus for positive energies the picture for local potential is quite similar to the free case.

This actually extends all the way to $V \in L^1$. Then the plot thickens. Wigner and von Neumann (the latter is my second most favorite mathematician) constructed a potential is $\sim \sin x/x$ as $x \to \infty$, but $H_V$ has a positive eigenvalue $\lambda = 1$. Moreover, the $L^\infty$ norm of the potential can be as small as you wish. This is already seriously amazing stuff. Classical particle will always escape to infinity if its energy is larger than maximum potential size. In quantum mechanics, you can stop it by arranging tiny bumps all the way to infinity but in a very specific pattern. Think of it as of stopping a freight train by placing ever smaller sand particles on the rail track. I should stress that this happens without friction - the total energy of the particle is conserved. If you allow your potential to decay slower than Coulomb rate $1/x$, the spectral phenomena get a lot richer yet. Eigenvalues can now be dense on $(0,\infty)$. Singular continuous spectrum may appear (I won't even go into details on what singular continuous spectrum is, but ask me if you are curious). I will only say that this is the least understood kind of the spectrum though it does appear in quite a few physically relevant models). But all the while, on top of all these eigenvalues, there is still underlying scattering structure. In particular, one can prove the following theorem (joint work with Michael Christ).

**Theorem 0.1.** *Assume that $V(x) \in L^p$, $p < 2$. Then for almost every (a.e.) $k > 0$, there exist two solutions $\psi_\pm(x,k)$ of the equation $H_V\psi = k^2\psi$ with asymptotic behavior*

$$\psi_\pm(x,k) = \exp\left(\pm ikx \mp \frac{i}{2k}\int_0^x V(y)\,dy\right)(1 + o(1)) \tag{1}$$

*as $x \to +\infty$.*

Observe the change from $L^1$ regime. Now we need an additional term in the asymptotics involving the integral of the potential (it is called WKB correction). Also, such solutions exist not for all $k$ but for almost every $k$. It is on the zero Lebesgue measure set where (1) fails that all sorts of exotic spectra can appear. The Theorem is also nearly sharp: there are examples of potentials $V(x) \notin L^2$ such that (1) fails for every energy. It turns out that the proof of Theorem 0.1 is linked with one of the most celebrated theorems in Fourier analysis: Carleson a.e. convergence theorem for the Fourier transform. It is common to define the Fourier transform of an $L^2$ function by defining it first on $L^1$ approximations and then passing to the limit in $L^2$. But one can ask a question: can we define it for $f \in L^2$ simply as

$$\hat{f}(k) = \lim_{N\to\infty} \int_{-N}^{N} \exp(ikx) f(x)\, dx?$$

The function is not $L^1$, so the integral may not converge for some $k$ due to resonance with $f$ (it is easy to construct such examples; think of $f(x) = \sin k_0 x / x!$). But resonance is unlikely to happen at too many frequencies. Is it true that the integral converges for a.e. $k$? The question is easier for $f \in L^p$, $1 < p < 2$, and has been solved by Zygmund in 1920s. The $p = 2$ case was known as Luzin's conjecture, and it remained open until 1950s when it has been resolved positively by Carleson. The solution involves quite subtle analysis, and the Caleson's theorem can be regarded as a sort of benchmark of a very hard result in analysis.

Theorem 0.1 is proved by considering a.e. convergence for more general integral operators than Fourier transform. One can build a series for $\psi_\pm(x,k)$ which consist of multilinear operators built out of blocks that look like $\int_x^\infty \exp(\pm 2ikx \mp \frac{i}{k}\int_0^x V(y)\, dy)$. The proof works for $V \in L^p$, $1 < p < 2$, but does not work for $p = 2$. Top Fourier analysis experts like Terry Tao and Christoff Thiele tried to think about this problem but so far it did not work out. The question whether Theorem 0.1 can be extended to $p = 2$ is sometimes called "nonlinear Carleson theorem". Stop by if you'd like to know why.

Here are some problems on Schrödinger that I have in mind.

1. The analog of Theorem 0.1 is completely open in dimensions higher than one. A conjecture by Barry Simon says that the $L^2$ condition in higher dimensions should be replaced by $\int_{\mathbb{R}^d} |V(x)|^2 (1+|x|)^{1-d}\, dx < \infty$. This is completely open. The best result for which scattering is known is $|V(x)| \leq C(1+|x|)^{-1-\epsilon}$.

2. I think that there are some undiscovered more general results about perturbations of self-adjoint operators behind this $L^2$ story in one dimension. If properly deciphered, they can perhaps be used to understand other phenomena in Schrödinger from unexpected angle. Here is one relevant question: let $A$ be self-adjoint operator with purely absolutely continuous (ac) spectrum, and $B$ compact self-adjoint operator. Let $\lambda_j$ be its eigenvalues. $B$ is called trace class if $\sum_j |\lambda_j| < \infty$, and it is called Hilbert-Schmidt if $\sum_j |\lambda_j|^2 < \infty$ (weaker condition). It is a classical result in spectral theory theory that if $B$ is trace class, then $A + B$ has the same ac spectrum as $A$. Other spectrum (eigenvalues) may appear, but ac spectrum does not change. There is also a theorem (due to Kuroda and von Neumann) that given operator $A$ with ac spectrum, you can find a perturbation $B$ from Hilbert-Schmidt class (actually, from anything weaker than trace class!) so that $A+B$ has purely point spectrum, and so ac spectrum disappears. The $L^2$ potential for Schrödinger is not trace class, only a Hilbert-Schmidt operator (almost - morally true for our purpose here), yet it cannot destroy ac spectrum. The proof of this fact is fairly magical to me. I can use but I do not really see the picture behind it. So the question is: given $A$ with ac spectrum, and $B$ from Hilbert-Schmidt class, can you find

general conditions under which $B$ will not destroy the ac spectrum of $A$? What would be these conditions? Something about commutation? What? It would be great to make progress here. I have a feeling that one needs to find a right notion, and the results will likely have influence on many other problems.