SOME RECENT RESULTS ON THE CRITICAL SURFACE QUASI-GEOSTROPHIC EQUATION: A REVIEW

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ABSTRACT. We review some recent results on the dissipative surface quasi-geostrophic equation, focusing on the critical case. We provide some background results and prove global existence of regular solutions.

1. INTRODUCTION

The $2D$ surface quasi-geostrophic equation attracted much attention lately from various authors (see e.g. [1, 2, 3, 5, 6, 8, 9, 12, 16, 20, 18, 22, 23, 26, 27] where more references can be found). Mainly it is due to the fact that this is probably the simplest evolutionary fluid dynamics equation for which the problem of existence of smooth global solutions remains unsolved. In this review we will consider the dissipative surface quasi-geostrophic (SQG) equation

$$\left\{ \begin{array}{l} \theta_t = u \cdot \nabla\theta - (-\Delta)^\alpha \theta, \;\; \theta(x,0) = \theta_0(x) \\ u = (u_1, u_2) = (-R_2\theta, R_1\theta). \end{array} \right. \tag{1}$$

Here $\theta : \mathbb{R}^2 \to \mathbb{R}$ is a scalar function, $\alpha > 0$, while R_1 and R_2 are the usual Riesz transforms: $(R_l \hat{f})(k) = \frac{-ik_l}{|k|}\widehat{f(k)}$. There are two natural settings for the equation: whole plane $\mathbb{R}^2$ with decaying initial data and torus $\mathbb{T}^2$ (or equivalently, periodic initial data in $\mathbb{R}^2$). In this review, we will focus on the periodic (torus) case.

The SQG equation can be derived via formal asymptotic expansion from the Boussinesq system for strongly rotating fluid in a half-space - a frequently used model for oceanic and atmospheric fluid flow (see e.g. [5], [24]). The function θ has a meaning of normalized temperature on the surface of the half-space. In mathematical literature, this equation appeared first in [8] (in the conservative case where there is no dissipative term). In particular, a blow up scenario (collapsing saddle) was identified in [8] and studied numerically. It was later shown that in this scenario, the blow up does not happen [12].

The equation (1) possesses a maximum principle: the L^p norms of the solution $\|\theta(x,t)\|_{L^p}$ are non-increasing, $1 \le p \le \infty$ ([25, 13]). That is the strongest general control of solution that has been known for (1) until recently. The $p = \infty$ maximum principle makes value $\alpha = 1/2$ critical.

It was well known for a while (see [9, 25]) that for $\alpha > \frac{1}{2}$ (the subcritical case), the initial value problem (1) with C^∞-smooth periodic initial data θ_0 has a global C^∞ solution. For more information about the properties of solutions in this regime, see for example [2, 17, 25].

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A significant amount of research focused specifically on the critical $\alpha = \frac{1}{2}$ case. The critical dissipative term $(-\Delta)^{1/2}\theta$ is physically relevant, modelling the Eckmann pumping effect in the boundary layer near the surface (see e.g. [5]). In particular, Constantin, Cordoba, and Wu in [6] showed that the global smooth solution exists provided that $\|\theta_0\|_\infty$ is small enough (see also [3] for a different choice of function space and [4, 18] for further extensions). Ju proved conditional regularity results involving geometric constraints [19]. Finally, in two independent works [22] and [1], it was proved that global smooth solutions exist for large initial data without additional assumptions of any kind. The paper [22] works in periodic setting and shows existence of smooth solutions for smooth initial data (the recent work [15] extended approach of [22] the whole space setting). The method of [22] is based on an elementary new idea: a nonlocal maximum principle. It shows that a certain modulus of continuity of the initial data is preserved by the evolution. Along with a simple rescaling procedure, this additional control is sufficient to show global regularity. We will review this proof below. The paper [1] follows a completely different plan. It proves that a certain class of weak solutions to the drift diffusion equation gain Hölder regularity starting from L^2 initial data, provided that the advection velocity satisfies uniform in time bound on its BMO norm. The proof is based on DiGiorgi-type iterative estimates.

Whether finite time blow up can happen for large initial data in the supercritical case $0 \leqslant \alpha < \frac{1}{2}$ remains completely open. For results on properties of local solutions, small initial data, and conditional regularity in the supercritical regime, see [3, 4, 9, 14, 16, 10, 11, 18, 20, 28, 29].

The goal of this review is to present several results on the properties of solutions of the critical SQG equation, starting from basic background to the global regularity proof of [22] and its corollaries. We start with proving local existence, uniqueness, and smoothening of solutions in Section 2. We consider the case of critical space initial data in Section 3. These results are not new, however our proofs do not seem to be in the literature for the SQG equation, and they are quite elementary. We discuss the nonlocal maximum principle and global existence of solutions in Section 4. Spacial analyticity is established in Section 5. The plan of this review, as well as proofs of most results, follow closely the recent paper [21], where the dissipative Burgers equation was considered. One section from [21] that we are unfortunately missing here is the section on the possible blow up in the supercritical case.

2. Existence, uniqueness and smoothening of solutions

In this and next section we review the basic questions on local existence, uniqueness and regularity of solutions. Most of the material presented here is known; see e.g. [14] for similar results proved using different methods.

Let us denote H^s the usual scale of Sobolev spaces on the torus $\mathbb{T}^2$, and $\|\cdot\|_s$ the corresponding norms. The main result of this section is the following Theorem.

Theorem 1. *Assume that initial data θ_0 belongs to H^s, $s > 1$. Then there exists $T(\|\theta_0\|_s) > 0$ and a solution $\theta(x,t)$ of (1) such that*

$$\theta(x,t) \in C([0,T], H^s) \cap L^2([0,T], H^{s+1/2}), \tag{2}$$

$$t^n \|\theta(\cdot,t)\|_{s+n/2} \leq C \tag{3}$$

for every $n \geq 0$. The solution $\theta(x,t)$ satisfying (2), (3) is unique.

Denote by P^N the orthogonal projection to the first $(2N+1)^2$ eigenfunctions of Laplacian, $e^{2\pi ikx}$, $k=(k_1,k_2)$, $|k_1|,|k_2|\le N$. Consider Galerkin approximations $\theta^N(x,t)$, satisfying

$$\theta^N_t = P^N(u^N\cdot\nabla\theta^N) - (-\Delta)^{1/2}\theta^N,\;\; \theta^N(x,0)=P^N\theta_0(x); \tag{4}$$

here $u^N=(-R_2\theta^N, R_1\theta^N)$. We start with deriving some a-priori bounds for the growth of Sobolev norms. Consider (4) on the Fourier side:

$$\hat\theta^N_t(k,t) = \pi\sum_{l+m=k,\;|l|,|m|,|k|\le N}\langle l,m^\perp\rangle\left(\frac{1}{|m|}-\frac{1}{|l|}\right)\hat\theta^N(l,t)\hat\theta^N(m,t) - (2\pi|k|)^s\hat\theta^N(k,t). \tag{5}$$

Here we symmetrized the first sum on the right hand side in l,m indexes, and $\langle l,m^\perp\rangle\equiv l_1m_2-l_2m_1$.

Lemma 1. *Assume that $s\ge0$ and $\beta\ge0$. Then*

$$\left|\int_{\mathbb{T}^2}(u^N\cdot\nabla)\theta^N(-\Delta)^s\theta^N\,dx\right| \le C\|\theta^N\|_q\|\theta^N\|^2_{s+\beta} \tag{6}$$

for any q satisfying $q>2-2\beta$.

Proof. According to (5), on the Fourier side, the integral in (6) is equal to (up to a constant factor)

$$\sum_{k+l+m=0,\;|k|,|l|,|m|\le N}\langle l,m^\perp\rangle\left(\frac{1}{|m|}-\frac{1}{|l|}\right)|k|^{2s}\hat\theta^N(k)\hat\theta^N(l)\hat\theta^N(m) =: S.$$

In what follows, we will omit the $|k|,|l|,|m|\le N$ condition from the summation. It is present throughout the proof of this lemma, in every sum. Symmetrizing, we obtain

$$\begin{aligned}|S| &= \frac13\left|\sum_{k+l+m=0}\left(\langle l,m^\perp\rangle\left(\frac{1}{|m|}-\frac{1}{|l|}\right)|k|^{2s}+\langle k,l^\perp\rangle\left(\frac{1}{|l|}-\frac{1}{|k|}\right)|m|^{2s}+\right.\right.\\ &\qquad\left.\left.\langle m,k^\perp\rangle\left(\frac{1}{|k|}\frac{1}{|m|}\right)|l|^{2s}\right)\hat\theta^N(k)\hat\theta^N(l)\hat\theta^N(m)\right|\le\\ &\quad 2\sum_{k+l+m=0,\;|l|\le|m|\le|k|}\left|\frac{\langle l,m^\perp\rangle(|l|-|m|)}{|m||l|}|k|^{2s}+\frac{\langle k,l^\perp\rangle(|k|-|l|)}{|k||l|}|m|^{2s}+\right.\\ &\qquad\left.\frac{\langle m,k^\perp\rangle(|m|-|k|)}{|m||k|}|l|^{2s}\right||\hat\theta^N(k)\hat\theta^N(l)\hat\theta^N(m)|.\end{aligned} \tag{7}$$

The factor in front of $|\hat\theta^N(k)\hat\theta^N(l)\hat\theta^N(m)|$ can be rewritten as

$$\left|\langle l,m^\perp\rangle\left(\frac{|k|^{2s}}{|m|}-\frac{|m|^{2s}}{|k|}+\frac{1}{|l|}(|m|^{2s}-|k|^{2s})+|l|^{2s}\left(\frac{1}{|k|}-\frac{1}{|m|}\right)\right)\right|. \tag{8}$$

Next, observe that under conditions $|l|\le|m|\le|k|,\;\; l+m+k=0$ as in (7), we have $|l|\le|k|/2,\;|m|\ge|k|/2$. Therefore, we can estimate (8) by

$$C|l||m|\left(|k|^{2s-1}+|l|^{2s+1}/(|m||k|)\right)\le C|l||m|^s|k|^s.$$

Coming back to (7), we see that

$$|S| \leq C \sum_{k+l+m=0} |l|^{1-2\beta}|m|^{s+\beta}|k|^{s+\beta}|\hat{\theta}^N(k)\hat{\theta}^N(a)\hat{\theta}^N(b)| \leq$$
$$C\|\theta^N\|^2_{s+\beta}\sum_l |l|^{1-2\beta}|\hat{\theta}^N(l)| \leq C\|\theta^N\|_q\|\theta^N\|^2_{s+\beta}.$$

Here the second inequality is due to Parseval and convolution estimate, and the third holds by Hölder's inequality for every $q > 2-2\beta$. □

Lemma 1 implies a differential inequality for the Sobolev norms of solutions of (4).

Lemma 2. *Assume that $q > 1$, and $s \geq 0$. Then*

$$\frac{d}{dt}\|\theta^N\|^2_s \leq C(q)\|\theta^N\|_q^{M(q,s)} - \|\theta^N\|^2_{s+1/2}. \tag{9}$$

If in addition $s = q$ then

$$\frac{d}{dt}\|\theta^N\|^2_s \leq C(\epsilon)\|\theta^N\|_s^{2+\frac{1}{2\epsilon}} - \|\theta^N\|^2_{s+1/2}, \tag{10}$$

for any

$$0 < \epsilon < \min\left(\frac{q-1}{2}, 1/2\right). \tag{11}$$

Proof. Multiplying both sides of (4) by $(-\Delta)^s\theta^N$, and applying Lemma 1, we obtain (here we put $\beta := 1/2-\epsilon$, with ϵ satisfying (11))

$$\frac{d}{dt}\|\theta^N\|^2_s \leq C(q,\epsilon,s)\|\theta^N\|_q\|\theta^N\|^2_{s+1/2-\epsilon} - 2\|\theta^N\|^2_{s+1/2}.$$

Observe that if $q \geq s+1/2-\epsilon$, the estimate (9) follows immediately. If $q < s+1/2-\epsilon$, by Hölder we obtain

$$\|\theta^N\|^2_{s+1/2-\epsilon} \leq \|\theta^N\|^{2(1-\delta)}_{s+1/2}\|\theta^N\|^{2\delta}_q \tag{12}$$

where $\delta = \dfrac{\epsilon}{s+1/2-q}$. Applying Young's inequality we finish the proof of (9) in this case.

The proof of (10) is similar. We have

$$\frac{d}{dt}\|\theta^N\|^2_s \leq C(s,\epsilon)\|\theta^N\|_s\|\theta^N\|^2_{s+1/2-\epsilon} - 2\|\theta^N\|^2_{s+1/2}.$$

Applying the estimate (12) with $q = s$ and $\delta = 2\epsilon$ and Young's inequality we obtain

$$\frac{d}{dt}\|\theta^N\|^2_s \leq C\|\theta^N\|_s^{1+4\epsilon}\|\theta^N\|^{2-4\epsilon}_{s+1/2} - 2\|\theta^N\|^2_{s+1/2} \leq C\|\theta^N\|_s^{2+\frac{1}{2\epsilon}} - \|\theta^N\|^2_{s+1/2}.$$

□

The following lemma is an immediate consequence of (10) and local existence of the solution to the differential equation $z' = Cz^{1+\frac{1}{4\epsilon}}$, $z(0) = z_0$.

Lemma 3. *Assume $s > 1$ and $\theta_0 \in H^s$. Then there exists time $T = T(s, \|\theta_0\|_s)$ such that for every N we have the bound (uniform in N)*

$$\|\theta^N\|_s(t) \leq C(s, \|\theta_0\|_s), \quad 0 \leq t \leq T, \tag{13}$$

Proof. From (10), we get that $z(t) \equiv \|\theta^N(\cdot,t)\|_s^2$ satisfies the differential inequality $z' \le Cz^{1+\frac{1}{4\epsilon}}$. This implies the bound (13) for time T which depends only on coefficients in the differential inequality and initial data. $\square$

Now, we obtain some uniform bounds for higher order H^s norms of the Galerkin approximations.

Lemma 4. *Assume $s > 1$ and $\theta_0 \in H^s$. Then there exists time $T = T(s, \|\theta_0\|_s)$ such that for every N we have the bounds (uniform in N)*

$$\int_0^T \|\theta^N(\cdot,t)\|_{s+1/2}^2\,dt < \frac{1}{2}\|\theta_0\|_s^2. \tag{14}$$

$$t^{n/2}\|\theta^N\|_{s+n/2} \le C(n,s,\|\theta_0\|_s), \ \ 0 < t \le T, \tag{15}$$

for any $n \ge 0$. Here time T is the same as in Lemma 3.

Proof. The inequality (14) follows from integrating (10) in time with T as in Lemma 3. We are going to first verify (15) by induction for positive integer n. For $n = 0$, the statement follows from Lemma 3. Inductively, assume that $\|\theta^N\|_{s+n/2}^2(t) \le Ct^{-n}$ for $0 \le t \le T$. Fix any $t \in (0,T]$, and consider the interval $I = (t/2, t)$. By (9) with s replaced by $s+n/2$ and q by s, we have for every $n \ge 0$

$$\frac{d}{dt}\|\theta^N\|_{s+n/2}^2 \le C\|\theta^N\|_s^M - \|\theta^N\|_{s+(n+1)/2}^2. \tag{16}$$

Due to Lemma 3 and our induction assumption,

$$\int_{t/2}^t \|\theta^N\|_{s+(n+1)/2}^2\,ds \le Ct + C\|\theta^N(t/2)\|_{s+n/2}^2 \le Ct^{-n}.$$

Thus we can find $\tau \in I$ such that

$$\|\theta^N(\tau)\|_{s+(n+1)/2}^2 \le C|I|^{-1}t^{-n} \le Ct^{-n-1}.$$

Moreover, from (16) with n changed to $n+1$ we find that

$$\|\theta^N(t)\|_{s+(n+1)/2}^2 \le \|\theta^N(\tau)\|_{s+(n+1)/2}^2 + Ct \le Ct^{-n-1},$$

concluding the proof for integer n. Non-integer n can be obtained by interpolation:

$$\|\theta^N\|_{s+r/2} \le \|\theta^N\|_s^{1-\frac{r}{n}}\|\theta^N\|_{s+n/2}^{\frac{r}{n}}, \quad 0 < r \le n.$$

$\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The proof of Theorem 1 is standard. It follows from (1) and (15) that for every small $\epsilon > 0$ and every $r > 0$ we have uniform in N and $t \in [\epsilon, T]$ bounds

$$\|\theta_t^N\|_r \le C(r,\epsilon). \tag{17}$$

By (15) and (17) and the well known compactness criteria (see e.g. [7], Chapter 8), we can find a subsequence θ^{N_j} converging in $C([\epsilon,T],H^r)$ to some function θ. Since ϵ and r are arbitrary one can apply the standard subsequence of subsequence procedure to find a subsequence (still denoted by θ^{N_j}) which converges to θ in $C((0,T],H^r)$, for any $r > 0$. The limiting function θ must satisfy the estimates (15) and it is straightforward to check that it solves the SQG equation on $(0,T]$. Thus, it remains to show that θ can be made to converge to θ_0 strongly in H^s as $t \to 0$.

We start by showing that θ converges to θ_0 as $t \to 0$ weakly in H^s. Let $\varphi(x)$ be an arbitrary C^∞ function. Consider

$$g^N(t,\varphi) \equiv (\theta^N, \varphi) = \int_{\mathbb{T}^2} \theta^N(x,t)\varphi(x)\,dx.$$

Clearly, $g^N(\cdot,\varphi) \in C([0,\tau])$, where $\tau \equiv T/2$. Also, taking inner product of (1) with φ we can show (due to L^2 boundedness of Riesz transforms) that for any $\delta > 0$,

$$\int_0^\tau |g_t^N|^{1+\delta}\,dt \le C\left(\int_0^\tau \|\theta^N\|_{L^2}^{2+2\delta}\|\varphi\|_{W^1_\infty}^{1+\delta}\,dt + \int_0^\tau \|\theta^N\|_{L^2}^{1+\delta}\|\varphi\|_1^{1+\delta}\,dt\right). \tag{18}$$

Due to the condition $s \ge 0$ and monotonicity of L^2 norm, we have that $\|\theta^N\|_{L^2} \le C$ on $[0,\tau]$, and thus $\|g_t^N(\cdot,\varphi)\|_{L^{1+\delta}} \le C(\varphi)$. Therefore the sequence $g^N(t,\varphi)$ is compact in $C([0,\tau])$, and we can pick a subsequence $g^{N_j}(t,\varphi)$ converging uniformly to a function $g(t,\varphi) \in C([0,\tau])$. Clearly, by choosing an appropriate subsequence we can assume $g(t,\varphi) = (\theta,\varphi)$ for $t \in (0,\tau]$. Next, we can choose a subsequence $\{N_j\}$ such that $g^{N_j}(t,\varphi)$ has a limit for any smooth function φ from a countable dense set in H^{-s}. Given that we have uniform control over $\|\theta^{N_j}\|_s$ on $[0,\tau]$, it follows that $g^{N_j}(t,\varphi)$ converges uniformly on $[0,\tau]$ for every $\varphi \in H^{-s}$. Now for any $t > 0$,

$$|(\theta-\theta_0,\varphi)| \le |(\theta-\theta^{N_j},\varphi)| + |(\theta^{N_j}-\theta_0^{N_j},\varphi)| + |(\theta_0^{N_j}-\theta_0,\varphi)|. \tag{19}$$

The first and the third terms on the right hand side of (19) can be made small uniformly in $(0,\tau]$ by choosing sufficiently large N_j. The second term tends to zero as $t \to 0$ for any fixed N_j. Thus $\theta(\cdot,t)$ converges to $\theta_0(\cdot)$ weakly in H^s as $t \to 0$. Consequently,

$$\|\theta_0(\cdot)\|_s \le \liminf_{t\to 0} \|\theta(\cdot,t)\|_s. \tag{20}$$

Furthermore, it follows from (10) that for every N the function $\|\theta^N\|_s^2(t)$ is always below the graph of the solution of the equation

$$z_t = Cz^{1+\frac{1}{4\epsilon}}, \quad z(0) = \|\theta_0\|_s^2.$$

By construction of the solution θ, the same is true for $\|\theta\|_s^2(t)$. Thus, $\|\theta_0\|_s \ge \limsup_{t\to 0} \|\theta\|_s(t)$. From this and (20), we obtain that $\|\theta_0\|_s = \lim_{t\to 0}\|\theta\|_s(t)$. This equality combined with weak convergence finishes the existence part of the proof.

We next turn to uniqueness. Assume that there is a second solution, v, with the same properties as θ. Denote by w the advection velocity corresponding to v. Then $f \equiv \theta - v$ satisfies

$$f_t = (u\cdot\nabla)f + ((u-w)\cdot\nabla)v - (-\Delta)^{1/2}f, \;\; f(0) = 0.$$

Taking inner product with f we obtain

$$\frac{1}{2}\partial_t\|f\|_{L^2} \le \int_{\mathbb{T}^2}(u\cdot\nabla)ff\,dx + \int_{\mathbb{T}^2}((u-w)\cdot\nabla)vf\,dx - \|f\|_{1/2}^2. \tag{21}$$

The first integral on the right hand side of (21) vanishes due to incompressibility of u. Let us estimate the second integral as follows:

$$\left|\int_{\mathbb{T}^2}((u-w)\cdot\nabla)vf\,dx\right| \le \|u-w\|_{L^{8/3}}\|f\|_{L^{8/3}}\|v\|_{W_4^1} \le C\|f\|^2_{L^{8/3}}\|v\|_{W_4^1} \le C\|f\|_{L^2}\|f\|_{1/2}\|v\|_{3/2}. \tag{22}$$

Here W_4^1 is the Sobolev space of L^4 functions with one derivative in L^4. We used Hölder inequality in the first step, boundedness of Riesz transform in $L^{8/3}$ in the second step, and Gagliardo-Nirenberg inequality and Sobolev imbedding in the last step. Putting (22) into (21) and applying Young's inequality, we obtain

$$\frac{1}{2}\partial_t\|f\|_{L^2} \le C\|f\|_{L^2}\|f\|_{1/2}\|v\|_{3/2} - \|f\|^2_{1/2} \le C'\|v\|^2_{3/2}\|f\|^2_{L^2}. \tag{23}$$

Recall that $v \in L^2([0,T], H^{s+1/2})$, where $s > 1$. Thus we can apply Gronwall yielding $\|f\|_{L^2} = 0$ for all $t \le T$. □

3. The case of the critical space H^1

Here we extend the result of Theorem 1 to the initial data θ_0 in the critical space H^1. Note that local existence of solutions with initial data in H^1 (and, more generally, in $H^{2-2\alpha}$ for the dissipation power $\alpha \in (0,1)$ in (1)) has been established in [23, 20] using different methods.

Theorem 2. *All the results of Theorem 1 remain valid for the initial data $\theta_0 \in H^1$, except the existence time T depends on θ_0 and not just $\|\theta_0\|_1$.*

Proof. We introduce the following Hilbert spaces of periodic functions. Let $\varphi : [0,\infty) \to [1,\infty)$ be an unbounded increasing function. Then $H^{s,\varphi}$ consists of periodic functions $f \in L^2$ such that its Fourier coefficients satisfy

$$\|f\|^2_{H^{s,\varphi}} := \sum_n |n|^{2s}\varphi(|n|)^2|\hat{f}(n)|^2 < \infty. \tag{24}$$

Note that $\theta_0 \in H^{1,\varphi}$ for some function φ. Without loss of generality we may assume, in addition, that $\varphi \in C^\infty$ and

$$\varphi'(x) \le Cx^{-1}\varphi(x) \tag{25}$$

for some constant C. It follows from (25) that

$$\varphi(2x) \le 2^C\varphi(x). \tag{26}$$

We start from Galerkin approximations. Consider the sum arising from the nonlinear term when estimating the H^s norm of the solution:

$$S := \sum_{l+m+k=0,|l|,|m|,|k|\le N} \langle l, m^\perp\rangle \left(\frac{1}{|m|} - \frac{1}{|l|}\right)|k|^{2s}\varphi(|k|)^2\hat{\theta}^N(l)\hat{\theta}^N(m)\hat{\theta}^N(k).$$

In what follows, for the sake of brevity, we will omit mentioning restrictions $|l|, |m|, |k| \le N$ in notation for the sums; all sums will be taken with this restriction. Observe that (cf. (7))

$$|S| \le 6 \sum_{k+l+m=0, |l| \le |m| \le |k|} \Big| \langle l, m^\perp \rangle \left(\frac{1}{|m|} - \frac{1}{|l|} \right) |k|^{2s} \varphi(|k|)^2 + \langle m, k^\perp \rangle \left(\frac{1}{|k|} - \frac{1}{|m|} \right) |l|^{2s} \varphi(|l|)^2$$

$$+ \langle k, l^\perp \rangle \left(\frac{1}{|l|} - \frac{1}{|k|} \right) |m|^{2s} \varphi(|m|)^2 \Big| |\hat{\theta}^N(k) \hat{\theta}^N(l) \hat{\theta}^N(m)|. \tag{27}$$

Recall that under conditions $|l| \le |m| \le |k|$, $l+m+k=0$, we have $|l| \le |k|/2$, $|m| \ge |k|/2$. Similarly to (7), the factor in (27) in front of $|\hat{\theta}^N(k)\hat{\theta}^N(l)\hat{\theta}^N(m)|$ does not exceed

(28)

$$\left| \langle l, m^\perp \rangle \left(\frac{|k|^{2s}\varphi(|k|)^2}{|m|} - \frac{|m|^{2s}\varphi(|m|)^2}{|k|} + \frac{|m|^{2s}\varphi(|m|)^2}{|l|} - \frac{|k|^{2s}\varphi(|k|)^2)}{|l|} + |l|^{2s}\varphi(|l|)^2 \frac{|m|-|k|}{|m||k|} \right) \right| \le$$
$$C|l||k|^s \varphi(|k|)^2 |m|^s + C|m| \left(|m|^{2s}(\varphi(|m|)^2 - \varphi(|k|)^2) + \varphi(|k|)^2(|m|^{2s} - |k|^{2s}) \right).$$

Using (25) and (26), we can further estimate the last line of (28) by

$$C|l||k|^s\varphi(|k|)|m|^s\varphi(|m|) \tag{29}$$

with a different constant C. Fix $M > 0$ to be specified later. Notice that sum over $|k| \le M$ in (27) can be bounded by a constant $C(M)$. Splitting summation in l over dyadic shells scaled with $|k|$, define

$$S_1(a) = \sum_{k+l+m=0, |b| \le |k|, |k| \ge M, |l| \in [2^{-a-1}|k|, 2^{-a}|k|]} |m|^{s+1/2}\varphi(|m|)|k|^{s+1/2}\varphi(|k|)|\hat{\theta}^N(k)\hat{\theta}^N(l)\hat{\theta}^N(m)|.$$

Then due to (29) and the relationship between l, m and k in the summation for S we have

$$|S| \le C \sum_{a=1}^{\infty} 2^{-a} S_1(a) + C(M). \tag{30}$$

Think of $S_1(a)$ as a quadratic form in $\hat{\theta}^N(k)$ and $\hat{\theta}^N(m)$. Then applying Schur test to each $S_1(a)$ we obtain

$$\begin{aligned} S_1(a) &\le \|\theta^N\|^2_{H^{s+1/2,\varphi}} \cdot \sup_{|k| \ge M} \sum_{|l| \in [2^{-a-1}|k|, 2^{-a}|k|]} |\hat{\theta}^N(l)| \le \\ & C\|\theta^N\|^2_{H^{s+1/2,\varphi}} \|\theta^N\|_{H^{1,\varphi}} (\varphi(2^{-a}M))^{-1}. \end{aligned} \tag{31}$$

Next, note that

$$\begin{aligned} \sum_{a=1}^{\infty} 2^{-a} S_1(a) &= \sum_{a=1}^{a_0} 2^{-a} S_1(a) + \sum_{a=a_0}^{\infty} 2^{-a} S_1(a) \le \\ & C\|\theta^N\|^2_{H^{s+1/2,\varphi}} \|\theta^N\|_{H^{1,\varphi}} \left(2^{1-a_0} + (\varphi(2^{-a_0}M))^{-1} \right). \end{aligned} \tag{32}$$

Given $\epsilon > 0$, we can choose, first, sufficiently large a_0 and then sufficiently large M to obtain from (30), (32) and unboundedness of φ

$$|S| \le C\epsilon \|\theta^N\|^2_{H^{s+1/2,\varphi}} \|\theta^N\|_{H^{1,\varphi}} + C(M(\epsilon)). \tag{33}$$

It follows from (4) and (33) that

$$\frac{d}{dt}\|\theta^N\|^2_{H^{s,\varphi}} \le (C\epsilon\|\theta^N\|_{H^{1,\varphi}} - 1)\|\theta^N\|^2_{H^{s+1/2,\varphi}} + C(\epsilon), \tag{34}$$

for all $s \ge 1$. Using this estimate and essentially the same arguments as before we can extend the results of Theorem 1 to the case $s = 1$. The only difference is that if $s > 1$ then the time of existence in Theorem 1 $T = T(\|\theta_0\|_s)$. If $s = 1$, then $\theta_0 \in H^{1,\varphi}$ for some function φ described at the beginning of the section and the existence time provided by the argument is not uniform in $\|\theta\|_1 : T = T(\theta_0) = T(\varphi, \|\theta_0\|_{H^{1,\varphi}})$. $\square$

4. Global regularity

In this section, we show that the solution described in Theorem 1 is in fact global. We will assume that the initial data θ_0 is C^∞. Due to Theorem 1 and its extension in Section 3, all results will hold for $\theta_0 \in H^1$ since solution corresponding to such initial data becomes smooth immediately. The main result is the following theorem.

Theorem 3. *The critical surface quasi-geostrophic equation with periodic smooth initial data $\theta_0(x)$ has a unique global smooth solution. Moreover, the following estimate holds for every time t:*

$$\|\nabla\theta(\cdot,t)\|_{L^\infty} \leqslant C\|\nabla\theta_0\|_{L^\infty} \exp\exp\{C\|\theta_0\|_{L^\infty}\}\,. \tag{35}$$

Proof. We follow the argument of [22].

The main idea is to show that critical surface quasi-geostrophic equation possesses a stronger maximum principle than L^∞ control. An interesting feature of this maximum principle is that it is nonlocal; it has the form of preservation of a certain family of moduli of continuity, sufficiently strong to allow control of $\|\nabla\theta\|_{L^\infty}$. Recall that a modulus of continuity is just an arbitrary increasing continuous concave function $\omega : [0,+\infty) \to [0,+\infty)$ such that $\omega(0) = 0$. Also, we say that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ has modulus of continuity ω if $|f(x) - f(y)| \leqslant \omega(|x-y|)$ for all $x, y \in \mathbb{R}^n$.

The flow term $u \cdot \nabla\theta$ in the dissipative quasi-geostrophic equation tends to make the modulus of continuity of θ worse while the dissipation term $(-\Delta)^{1/2}\theta$ tends to make it better. Our aim is to construct some special moduli of continuity for which the dissipation term always prevails and such that every periodic C^∞-function θ_0 has one of these special moduli of continuity.

Note that the critical SQG equation has a simple scaling invariance: if $\theta(x,t)$ is a solution, then so is $\theta(Bx, Bt)$. This means that if we manage to find one modulus of continuity ω that is preserved by the dissipative evolution for **all** periodic solutions (i.e., with arbitrary lengths and spacial orientations of the periods), then the whole family $\omega_B(\xi) = \omega(B\xi)$ of moduli of continuity will also be preserved for all periodic solutions.

Observe now that if ω is unbounded, then any given C^∞ periodic function has modulus of continuity ω_B if $B > 0$ is sufficiently large. Also, if the modulus of continuity ω has finite derivative at 0, it can be used to estimate $\|\nabla\theta\|_\infty$. Thus, our task reduces to constructing an unbounded modulus of continuity with finite derivative at 0 that is preserved by the critical SQG evolution.

From now on, we will also assume that, in addition to unboundedness and the condition $\omega'(0) < +\infty$, we have $\lim_{\xi\to 0+} \omega''(\xi) = -\infty$. Then, if a C^∞ periodic function f has modulus of continuity ω, we have

$$\|\nabla f\|_\infty < \omega'(0)\,.$$

Indeed, take a point $x \in \mathbb{R}^2$ at which $\max|\nabla f|$ is attained and consider the point $y = x + \xi e$ where $e = \frac{\nabla f}{|\nabla f|}$. Then we must have $f(y) - f(x) \leqslant \omega(\xi)$ for all $\xi \geqslant 0$. But the left hand side is at least $|\nabla f(x)|\xi - C\xi^2$ where $C = \frac{1}{2}\|\nabla^2 f\|_\infty$ while the right hand side can be represented as $\omega'(0)\xi - \rho(\xi)\xi^2$ with $\rho(\xi) \to +\infty$ as $\xi \to 0+$. Thus $|\nabla f(x)| \leqslant \omega'(0) - (\rho(\xi) - C)\xi$ for all $\xi > 0$ and it remains to choose some $\xi > 0$ satisfying $\rho(\xi) > C$.

Now assume that θ has modulus of continuity ω for all times $t < t_0$. Then θ remains C^∞ smooth up to t_0 (see Appendix II) and, according to the local regularity theorem, for a short time beyond t_0. By continuity, we see that θ must also have modulus of continuity ω at the moment t_0. Suppose that $|\theta(x,t_0) - \theta(y,t_0)| < \omega(|x-y|)$ for all $x \neq y$. We claim that then θ has modulus of continuity ω for all $t > t_0$ sufficiently close to t_0. Indeed, by the remark above, at the moment t_0 we have $\|\nabla\theta\|_\infty < \omega'(0)$. By continuity of derivatives, this also holds for $t > t_0$ close to t_0, which immediately takes care of the inequality $|\theta(x,t) - \theta(y,t)| < \omega(|x-y|)$ for small $|x-y|$. Also, since ω is unbounded and $\|\theta\|_\infty$ doesn't grow with time, we automatically have $|\theta(x,t) - \theta(y,t)| < \omega(|x-y|)$ for large $|x-y|$. The last observation is that, due to periodicity of θ, it suffices to check the inequality $|\theta(x,t) - \theta(y,t)| < \omega(|x-y|)$ for x belonging to some compact set $K \subset \mathbb{R}^2$. Thus, we are left with the task to show that, if $|\theta(x,t_0) - \theta(y,t_0)| < \omega(|x-y|)$ for all $x \in K$, $\delta \leqslant |x-y| \leqslant \delta^{-1}$ with some fixed $\delta > 0$, then the same inequality remains true for a short time beyond t_0. But this immediately follows from the uniform continuity of θ.

This implies that the only scenario in which the modulus of continuity ω may be lost by θ is the one in which there exists a moment $t_0 > 0$ such that θ has modulus of continuity ω for all $t \in [0, t_0]$ and there are two points $x \neq y$ such that $\theta(x,t_0) - \theta(y,t_0) = \omega(|x-y|)$. We shall rule this scenario out by showing that, in such case, the derivative $\frac{\partial}{\partial t}(\theta(x,t) - \theta(y,t))\big|_{t=t_0}$ must be negative, which, clearly, contradicts the assumption that the modulus of continuity ω is preserved up to the time t_0.

Before we start the actual estimate of different terms at time t_0, we need the following lemma to relate regularity of θ and u. Singular integral operators like Riesz transforms appearing in (1) do not preserve moduli of continuity in general but they do not spoil them too much either. More precisely, we have

Lemma 5. *If the function θ has modulus of continuity ω, then $u = (-R_2\theta, R_1\theta)$ has modulus of continuity*

$$\Omega(\xi) = A\left(\int_0^\xi \frac{\omega(\eta)}{\eta}\,d\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,d\eta\right) \tag{36}$$

with some universal constant $A > 0$.

The proof of this result is elementary. To make the paper self-contained, we provide a sketch of it in the Appendix I.

Assume that the above breakthrough scenario takes place. Let $\xi = |x-y|$. Observe that $(u \cdot \nabla\theta)(x) = \frac{d}{dh}\theta(x + hu(x))\big|_{h=0}$ and similarly for y. But

$$\theta(x + hu(x)) - \theta(y + hu(y)) \leqslant \omega(|x-y| + h|u(x) - u(y)|) \leqslant \omega(\xi + h\Omega(\xi))$$

where Ω is given by (36). Since $\theta(x)-\theta(y)=\omega(\xi)$, we conclude that

$$(u\cdot\nabla\theta)(x)-(u\cdot\nabla\theta)(y)\leqslant\Omega(\xi)\omega'(\xi)\,.$$

Consider now the dissipative term. Recall that it can be written as $\frac{d}{dh}\mathcal{P}_h*\theta\big|_{h=0}$ where $\mathcal{P}_h$ is the usual Poisson kernel in $\mathbb{R}^2$ (again, this formula holds for all smooth periodic functions regardless of the lengths and spatial orientation of the periods, which allows us to freely use the scaling and rotation tricks below). Thus, our task is to estimate $(\mathcal{P}_h*\theta)(x)-(\mathcal{P}_h*\theta)(y)$ under the assumption that θ has modulus of continuity ω. Since everything is translation and rotation invariant, we may assume that $x=(\frac{\xi}{2},0)$ and $y=(-\frac{\xi}{2},0)$.

Write

$$\begin{aligned}(\mathcal{P}_h*\theta)(x)-(\mathcal{P}_h*\theta)(y)&=\iint_{\mathbb{R}^2}[\mathcal{P}_h(\tfrac{\xi}{2}-\eta,\nu)-\mathcal{P}_h(-\tfrac{\xi}{2}-\eta,\nu)]\theta(\eta,\nu)\,d\eta d\nu\\
&=\int_{\mathbb{R}}d\nu\int_0^\infty[\mathcal{P}_h(\tfrac{\xi}{2}-\eta,\nu)-\mathcal{P}_h(-\tfrac{\xi}{2}-\eta,\nu)][\theta(\eta,\nu)-\theta(-\eta,\nu)]\,d\eta\\
&\leqslant\int_{\mathbb{R}}d\nu\int_0^\infty[\mathcal{P}_h(\tfrac{\xi}{2}-\eta,\nu)-\mathcal{P}_h(-\tfrac{\xi}{2}-\eta,\nu)]\omega(2\eta)\,d\eta\\
&=\int_0^\infty[P_h(\tfrac{\xi}{2}-\eta)-P_h(-\tfrac{\xi}{2}-\eta)]\omega(2\eta)\,d\eta\\
&=\int_0^\xi P_h(\tfrac{\xi}{2}-\eta)\omega(2\eta)\,d\eta+\int_0^\infty P_h(\tfrac{\xi}{2}+\eta)[\omega(2\eta+2\xi)-\omega(2\eta)]\,d\eta\end{aligned}$$

where P_h is the 1-dimensional Poisson kernel. Here we used symmetry and monotonicity of the Poisson kernels together with the observation that $\int_{\mathbb{R}}\mathcal{P}_h(\eta,\nu)\,d\nu=P_h(\eta)$. The last formula can also be rewritten as

$$\int_0^{\frac{\xi}{2}}P_h(\eta)[\omega(\xi+2\eta)+\omega(\xi-2\eta)]\,d\eta+\int_{\frac{\xi}{2}}^\infty P_h(\eta)[\omega(2\eta+\xi)-\omega(2\eta-\xi)]\,d\eta\,.$$

Recalling that $\int_0^\infty P_h(\eta)\,d\eta=\frac{1}{2}$, we see that the difference $(\mathcal{P}_h*\theta)(x)-(\mathcal{P}_h*\theta)(y)-\omega(\xi)$ can be estimated from above by

$$\begin{aligned}&\int_0^{\frac{\xi}{2}}P_h(\eta)[\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)]\,d\eta\\
&+\int_{\frac{\xi}{2}}^\infty P_h(\eta)[\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)]\,d\eta\,.\end{aligned}$$

Recalling the explicit formula for P_h, dividing by h and passing to the limit as $h\to0+$, we finally conclude that the contribution of the dissipative term to our derivative is bounded from above by

$$\begin{aligned}&\frac{1}{\pi}\int_0^{\frac{\xi}{2}}\frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}\,d\eta\\
&+\frac{1}{\pi}\int_{\frac{\xi}{2}}^\infty\frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^2}\,d\eta\,.\end{aligned}\tag{37}$$

Note that due to concavity of ω, both terms are strictly negative.

We will now construct our special modulus of continuity as follows. Choose two small positive numbers $\delta > \gamma > 0$ and define the continuous function ω by

$$\omega(\xi) = \xi - \xi^{\frac{3}{2}} \qquad \text{when } 0 \leqslant \xi \leqslant \delta$$

and

$$\omega'(\xi) = \frac{\gamma}{\xi(4 + \log(\xi/\delta))} \qquad \text{when } \xi > \delta\,.$$

Note that, for small δ, the left derivative of ω at δ is about 1 while the right derivative equals $\frac{\gamma}{4\delta} < \frac{1}{4}$. So ω is concave if δ is small enough. It is clear that $\omega'(0) = 1$, $\lim_{\xi \to 0+} \omega''(\xi) = -\infty$ and that ω is unbounded (it grows at infinity like double logarithm). The hard part, of course, is to show that, for this ω, the negative contribution to the time derivative coming from the dissipative term prevails over the positive contribution coming from the flow term. More precisely, we have to check the inequality

$$A\left[\int_0^\xi \frac{\omega(\eta)}{\eta}\,d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,d\eta\right]\omega'(\xi) + \frac{1}{\pi}\int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta) + \omega(\xi-2\eta) - 2\omega(\xi)}{\eta^2}\,d\eta$$
$$+\frac{1}{\pi}\int_{\frac{\xi}{2}}^\infty \frac{\omega(2\eta+\xi) - \omega(2\eta-\xi) - 2\omega(\xi)}{\eta^2}\,d\eta < 0 \qquad \text{for all } \xi > 0\,.$$

Let $0 \leqslant \xi \leqslant \delta$. Since $\omega(\eta) \leqslant \eta$ for all $\eta \geqslant 0$, we have $\int_0^\xi \frac{\omega(\eta)}{\eta}\,d\eta \leqslant \xi$ and $\int_\xi^\delta \frac{\omega(\eta)}{\eta^2}\,d\eta \leqslant \log\frac{\delta}{\xi}$. Now,

$$\int_\delta^\infty \frac{\omega(\eta)}{\eta^2}\,d\eta = \frac{\omega(\delta)}{\delta} + \gamma\int_\delta^\infty \frac{1}{\eta^2(4+\log(\eta/\delta))}\,d\eta \leqslant 1 + \frac{\gamma}{4\delta} < 2\,.$$

Observing that $\omega'(\xi) \leqslant 1$, we conclude that the positive part of the left hand side is bounded by $A\xi(3 + \log\frac{\delta}{\xi})$.

To estimate the negative part, we just use the first integral in (37). Note that $\omega(\xi+2\eta) \leqslant \omega(\xi) + 2\omega'(\xi)\eta$ due to concavity of ω, and $\omega(\xi-2\eta) \leqslant \omega(\xi) - 2\omega'(\xi)\eta - 2\omega''(\xi)\eta^2$ due to the second order Taylor formula and monotonicity of ω'' on $[0,\xi]$. Plugging these inequalities into the integral, we get the bound

$$\frac{1}{\pi}\int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}\,d\eta \leqslant \frac{1}{\pi}\xi\omega''(\xi) = -\frac{3}{4\pi}\xi\xi^{-\frac{1}{2}}\,.$$

But, obviously, $\xi\left(A(3+\log\frac{\delta}{\xi}) - \frac{3}{4\pi}\xi^{-\frac{1}{2}}\right) < 0$ on $(0,\delta]$ if δ is small enough.

Now let $\xi \geqslant \delta$. In this case, we have $\omega(\eta) \leqslant \eta$ for $0 \leqslant \eta \leqslant \delta$ and $\omega(\eta) \leqslant \omega(\xi)$ for $\delta \leqslant \eta \leqslant \xi$. Hence

$$\int_0^\xi \frac{\omega(\eta)}{\eta}\,d\eta \leqslant \delta + \omega(\xi)\log\frac{\xi}{\delta} \leqslant \omega(\xi)\left(2 + \log\frac{\xi}{\delta}\right)$$

because $\omega(\xi) \geqslant \omega(\delta) > \frac{\delta}{2}$ if δ is small enough.

Also

$$\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,d\eta = \frac{\omega(\xi)}{\xi} + \gamma\int_\xi^\infty \frac{d\eta}{\eta^2(4+\log(\eta/\delta))} \leqslant \frac{\omega(\xi)}{\xi} + \frac{\gamma}{\xi} \leqslant \frac{2\omega(\xi)}{\xi}$$

if $\gamma < \frac{\delta}{2}$ and δ is small enough.

Thus, the positive term on the left hand side is bounded from above by the expression $A\omega(\xi)\left(4+\log\frac{\xi}{\delta}\right)\omega'(\xi)=A\gamma\frac{\omega(\xi)}{\xi}$.

To estimate the negative term, note that, for $\xi\geqslant\delta$, we have

$$\omega(2\xi)\leqslant\omega(\xi)+\frac{\gamma}{4}\leqslant\frac{3}{2}\omega(\xi)$$

under the same assumptions on γ and δ as above. Also, due to concavity, we have $\omega(2\eta+\xi)-\omega(2\eta-\xi)\leqslant\omega(2\xi)$ for all $\eta\geqslant\frac{\xi}{2}$. Therefore,

$$\frac{1}{\pi}\int_{\frac{\xi}{2}}^{\infty}\frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^2}\,d\eta\leqslant-\frac{1}{2\pi}\int_{\frac{\xi}{2}}^{\infty}\frac{\omega(\xi)}{\eta^2}\,d\eta=-\frac{1}{\pi}\frac{\omega(\xi)}{\xi}\,.$$

But $\frac{\omega(\xi)}{\xi}(A\gamma-\frac{1}{\pi})<0$ if γ is small enough. This proves that the breakthrough scenario is impossible. The estimate (35) is straightforward to obtain using the behavior of $\omega(\xi)$ as $\xi\to\infty$.

Finally, if we have uniform control of $\|\nabla\theta\|_{L^\infty}$, then standard methods yield global existence of solutions and uniform in time bounds for all H^s norms. For the sake of completeness, we sketch this argument in Appendix II. □

5. Analyticity

Here, we show that global smooth solution guaranteed by Theorem 3 is analytic in spacial variables.

Theorem 4. *Assume that the initial data $\theta_0\in H^1$. Then the unique global solution of the critical SQG equation guaranteed to exist by Theorems 2 and 3 is real analytic for any $t>0$.*

Proof. Without loss of generality, we will assume that the initial data $\theta_0\in H^3$. Even if we started from θ_0 which is only in H^1, Theorem 2 implies that we gain the desired smoothness immediately.

Let us recall the Fourier representation of the Galerkin approximations to the critical SQG equation:

$$\hat\theta^N_t(k,t)=\pi\sum_{l+m=k,\ |l|,|m|,|k|\le N}\langle l,m^\perp\rangle\left(\frac{1}{|m|}-\frac{1}{|l|}\right)\hat\theta^N(l,t)\hat\theta^N(m,t)-2\pi|k|\hat\theta^N(k,t). \tag{38}$$

To simplify notation we will henceforth omit the restrictions $|l|,|m|,|k|\le N$ in any summation, but they are always present in the remainder of the proof. Put $\xi^N_k(t):=\hat\theta^N(k,t)e^{\pi|k|t}$. Observe that since $\theta(x,t)$ is real, $\overline{\xi}^N_k=\xi^N_{-k}$. We have

$$\xi^N_t(k,t)=\pi\sum_{l+m=k,\ |l|,|m|,|k|\le N}e^{-\gamma_{l,m,k}t}\langle l,m^\perp\rangle\left(\frac{1}{|m|}-\frac{1}{|l|}\right)\xi^N(l,t)\xi^N(m,t)-\pi|k|\xi^N(k,t), \tag{39}$$

where $\gamma_{l,m,k}:=\frac{1}{2}(|l|+|m|-|k|)$. Note that

$$0\le\gamma_{l,m,k}\le\min\{|l|,\ |m|\}. \tag{40}$$

Consider $Y_N(t) := \sum_k |k|^6|\xi_k^N(t)|^2$. Then we have

$$\begin{aligned}
\frac{dY_N}{dt} &= \Re\left(2\sum_{l+m+k=0} e^{-\gamma_{l,m,k}t}\langle l, m^\perp\rangle\left(\frac{1}{|m|}-\frac{1}{|l|}\right)|k|^6\xi_l^N\xi_m^N\xi_k^N\right) - \sum_k |k|^7|\xi_k^N|^2 \\
&= \Re\left(2\sum_{l+m+k=0}\langle l, m^\perp\rangle\left(\frac{1}{|m|}-\frac{1}{|l|}\right)|k|^6\xi_l^N\xi_m^N\xi_k^N\right) \\
&+ \Re\left(2\sum_{l+m+k=0}(e^{-\gamma_{l,m,k}t}-1)\langle l, m^\perp\rangle\left(\frac{1}{|m|}-\frac{1}{|l|}\right)|k|^6\xi_l^N\xi_m^N\xi_k^N\right) \\
&- \sum_k |k|^7|\xi_k^N|^2 =: I_1 + I_2 + I_3.
\end{aligned} \tag{41}$$

Symmetrizing I_1 over l, m and k we obtain

$$|I_1| = \frac{2}{3}\left|\Re\left(\sum_{l+m+k=0}\left(\langle l, m^\perp\rangle\left(\frac{1}{|m|}-\frac{1}{|l|}\right)|k|^6 + \langle k, l^\perp\rangle\left(\frac{1}{|l|}-\frac{1}{|k|}\right)|m|^6 + \langle m, k^\perp\rangle\left(\frac{1}{|k|}-\frac{1}{|m|}\right)|l|^6\right)\xi_l^N\xi_m^N\xi_k^N\right)\right| \leq 2\sum_{l+m+k=0,\ |l|\leq|m|\leq|k|}\left|\langle l, m^\perp\rangle\left(\frac{1}{|m|}-\frac{1}{|l|}\right)|k|^6 + \langle k, l^\perp\rangle\left(\frac{1}{|l|}-\frac{1}{|k|}\right)|m|^6 + \langle m, k^\perp\rangle\left(\frac{1}{|k|}-\frac{1}{|m|}\right)|l|^6\right||\xi_l^N\xi_m^N\xi_k^N|.$$

Similarly to (8) and argument right after it, we can show that

$$|I_1| \leq C\sum_{l+m+k=0,\ |l|\leq|m|\leq|k|} |l||m|^3|k|^3|\xi_l^N||\xi_m^N||\xi_k^N| \leq CY_N\sum |l||\xi_l^N| \leq CY_N^{3/2}. \tag{42}$$

Here in the second step we used convolution inequality and in the last step we used Hölder inequality:

$$\sum_l |l||\xi_l^N| \leq \left(\sum_{l\neq 0}|l|^{-3}\right)^{1/2} Y_N^{1/2}(t). \tag{43}$$

Observe that if $l+m+k=0$, then

$$\left|\langle l, m^\perp\rangle\left(\frac{1}{|m|}-\frac{1}{|l|}\right)\right| \leq |k|,$$

and hence for I_2 we have

$$|I_2| \leq 2\sum_{l+m+k=0}\min(|l|,|m|)t|k|^7|\xi_l^N||\xi_m^N||\xi_k^N|.$$

Here we used (40). Furthermore,

$$\sum_{l+m+k=0} \min(|l|,|m|)|k|^7|\xi_l^N||\xi_m^N||\xi_k^N| \le C\left(\sum_{l+m+k=0,\ |l|\le|m|\le|k|} |l||k|^7|\xi_l^N||\xi_m^N||\xi_k^N| + \right.$$

$$\left.\sum_{l+m+k=0,\ |m|\le|l|\le|k|} |m||k|^7|\xi_l^N||\xi_m^N||\xi_k^N|\right) \le C \sum_{l+m+k=0} |l||m|^{7/2}|k|^{7/2}|\xi_l^N||\xi_m^N||\xi_k^N|$$

$$\le C\left(\sum_l |l||\xi_l^N|\right)\left(\sum_k |k|^7|\xi_k^N|^2\right).$$

We used Young's inequality for convolution in the last step. Combining all estimates and applying (43), we obtain

$$|I_2| \le CtY_N^{1/2}\sum_k |k|^7|\xi_k^N|^2. \tag{44}$$

Combining (41), (42) and (44) we arrive at

$$\frac{dY_N}{dt} \le C_1 Y_N^{3/2} + (C_2 Y_N^{1/2} t - 1)\sum_k |k|^7|\xi_k^N|^2. \tag{45}$$

Note that $Y_N(0) = \|\theta_0^N\|_3^2$. Thus we have a differential inequality for Y_N ensuring upper bound on Y_N uniform in N for a short time interval τ which depends only on $\|\theta_0\|_3$. Observe that Lemma 6 in the Appendix II below ensures that the H^3 norm of solution $\theta(x,t)$ is bounded uniformly on $[0,\infty)$. Thus we can use the above construction to prove for every $t > 0$ uniform in N bounds on $\sum_k |\hat\theta^N(k,t)|^2 e^{\delta|k|}$ for some small $\delta(t, \|\theta_0\|_3) > 0$; δ is bounded away from zero if t is in any compact set of $(0,\infty)$. By construction of θ, it must satisfy the same bound. $\square$

6. Appendix I

Here we provide a sketch of the proof of Lemma 5.

Proof. The Riesz transforms are singular integral operators with kernels $K(r,\zeta) = r^{-2}\Omega(\zeta)$, where (r,ζ) are the polar coordinates. The function Ω is smooth and $\int_{S^1}\Omega(\zeta)d\sigma(\zeta) = 0$. Assume that the function f satisfies $|f(x) - f(y)| \le \omega(|x-y|)$ for some modulus of continuity ω. Take any x, y with $|x-y| = \xi$, and consider the difference

$$P.V.\int K(x-t)f(t)\,dt - P.V.\int K(y-t)f(t)\,dt \tag{46}$$

with integrals understood in the principal value sense. Note that

$$\left|P.V.\int_{|x-t|\le 2\xi} K(x-t)f(t)\,dt\right| = \left|P.V.\int_{|x-t|\le 2\xi} K(x-t)(f(t)-f(x))\,dt\right| \le C\int_0^{2\xi}\frac{\omega(r)}{r}\,dr.$$

Since ω is concave, we have

$$\int_0^{2\xi}\frac{\omega(r)}{r}\,dr \le 2\int_0^{\xi}\frac{\omega(r)}{r}\,dr.$$

A similar estimate holds for the second integral in (46). Next, let $\tilde{x} = \frac{x+y}{2}$. Then

$$\left|\int_{|x-t|\geq 2\xi} K(x-t)f(t)\,dt - \int_{|y-t|\geq 2\xi} K(y-t)f(t)\,dt\right| =$$

$$\left|\int_{|x-t|\geq 2\xi} K(x-t)(f(t)-f(\tilde{x}))\,dt - \int_{|y-t|\geq 2\xi} K(y-t)(f(t)-f(\tilde{x}))\,dt\right|$$

$$\leq \int_{|\tilde{x}-t|\geq 3\xi} |K(x-t)-K(y-t)||f(t)-f(\tilde{x})|\,dt +$$

$$\int_{3\xi/2\leq|\tilde{x}-t|\leq 3\xi} (|K(x-t)|+|K(y-t)|)|f(t)-f(\tilde{x})|\,dt.$$

Since

$$|K(x-t)-K(y-t)| \leq C\frac{|x-y|}{|\tilde{x}-t|^3}$$

when $|\tilde{x}-t|\geq 3\xi$, the first integral is estimated by $C\xi\int_{3\xi}^{\infty}\frac{\omega(r)}{r^2}\,dr$. The second integral is estimated by $C\omega(3\xi)$, and hence is controlled by $3C\int_0^{\xi}\frac{\omega(r)}{r}\,dr$. $\square$

7. Appendix II

Theorem 1 gives us local existence of smooth solution $\theta(x,t)$. The proof of Theorem 3 shows that $\|\nabla\theta\|_{L^\infty}$ remains uniformly bounded in time. Here we show that in this case, the higher order Sobolev norms of the solution also remain uniformly bounded.

Lemma 6. *Let $\theta(x,t)$ be a smooth solution of* (1)*. Assume that for every $0\leq t\leq T$, we have $\|\nabla\theta(\cdot,t)\|_{L^\infty}\leq C<\infty$. Then for every $s>0$ and every $0<t\leq T$, we also have $\|\theta(\cdot,t)\|_s\leq C(s)$.*

Proof. Let us denote by $|D^l f(x)|$ the sum of absolute values of all partial derivatives of order l of f at the point x. Consider the estimate for the H^s norm of the solution:

$$\frac{1}{2}\partial_t\|\theta\|_s^2 \leq \left|\int_{\mathbb{T}^2}(u\cdot\nabla)\theta(-\Delta)^s\theta\,dx\right| - \|\theta\|_{s+1/2}^2. \tag{47}$$

Without loss of generality, we can assume that s is an integer greater than 1. Integrating by parts in the integral on the right hand side of (47) and using incompressibility, we obtain that this integral is bounded by

$$C\sum_{l=1}^{s}\int_{\mathbb{T}^2}|D^l u||D^{s-l+1}\theta||D^s\theta|\,dx. \tag{48}$$

Let us estimate the first term in the sum (48); the rest is similar. We have

$$\int_{\mathbb{T}^2}|Du||D^s\theta|^2\,dx \leq \|D\theta\|_{L^3}\|D^s\theta\|_{L^3}^2 \leq C\|D\theta\|_{L^3}\|\theta\|_s^{2/3}\|\theta\|_{s+1/2}^{4/3}$$

$$\leq C\|D\theta\|_{L^3}\|\theta\|_1^{\frac{2}{3(2s-1)}}\|\theta\|_{s+1/2}^{2-\frac{2}{3(2s-1)}}. \tag{49}$$

Here in the first step we used Hölder inequality and boundedness of Riesz transform in L^3, in the second step we used fractional Sobolev imbedding and Hölder inequality, and in the last step Hölder inequality again. Since $\|\nabla\theta\|_{L^\infty}$ is uniformly bounded, we see that due to (49),

$$\frac{1}{2}\partial_t\|\theta\|_s^2 \le C_1\|\theta\|_{s+1/2}^{2-\frac{1}{3(s-1/2)}}\left(C_2 - \|\theta\|_{s+1/2}^{\frac{1}{3(s-1/2)}}\right). \tag{50}$$

Clearly, (50) implies the result of the lemma. □

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