Enhancement of diffusion by fluid flow. When people pour cream into coffee, few have patience to wait while diffusion produces a uniform mixture for them. Most use fluid flow - a swirl with a spoon - to speed up the process. Enhancement of diffusion by fluid flow is a phenomenon important in many processes in nature and engineering, from reactions in atmosphere to combustion engines and fecundity of marine organisms. My work in this direction started from conversations with astrophysicists at the University of Chicago who were interested in testing parts of their complex code developed for modeling nuclear combustion in stars on some well understood simpler equations. The fluid motion plays an important role in mixing the reactants and speeding up the combustion process. From the mathematical point of view, the problem turned out to be a wonderful mix of partial differential equations, functional analysis and dynamical systems.

Let us consider

$$\phi^A_t + A(u\cdot\nabla)\phi^A - \Delta\phi^A = 0, \;\; \phi^A(x,0) = \phi_0(x) \tag{1}$$

on a finite compact smooth manifold Ω. If we did not have a flow, the solution of heat equation would converge to its mean as time goes to infinity. The question is which flows can help it happen dramatically faster. Namely, consider the following definition.

Definition 0.1. *A C^1 flow u is called relaxation enhancing (RE) if for every $\tau > 0, \delta > 0$ there exists $A(\tau,\delta)$ such that if $A > A(\tau,\delta)$ then $\|\phi^A(\cdot,\tau) - \overline{\phi}_0\|_{L^2} \leq \delta$.*

In other words, our solution can be brought arbitrarily close to the mean value in arbitrary short time if only the flow coupling is strong enough. It turns out there is a sharp condition describing what flows are RE.

Theorem 0.2. *A C^1 flow u is RE if and only if the operator $u\cdot\nabla$ has no eigenfunctions in $H^1(\Omega)$.*

Here $H^1(\Omega)$ is the usual Sobolev space of functions that have first order derivatives in L^2. In other words, a flow is RE if $u\cdot\nabla$ does not have regular eigenfunctions. In particular the flow is RE if the spectrum of $u\cdot\nabla$ is purely continuous: it has no L^2 eigenfunctions at all. Smooth flows like that exist, and they were constructed first by Kolmogorov. In the theory of dynamical systems, such flows are called weakly mixing (stronger than ergodic but weaker than mixing). Are these hard to construct? Not extremely hard but not easy. They should look very wiggly. It should be possible to get a picture from Matlab - but my Matlab skills are not that great yet. Someone would be most welcome to help!

It turns out that the question can be recast in a much more general form. Assume that $\Gamma \geq 0$ is a non-negative self-adjoint operator in a Hilbert space H, but perhaps with a big kernel. Let L be another self-adjoint operator. Consider equation

$$\partial_t\psi^A = iAL\psi^A - \Gamma\psi^A, \;\; \psi^A(0) = \psi_0. \tag{2}$$

with A being a coupling constant. If we had only iAL part in the equation above, the L^2 norm of the solution would preserved, dynamics will be unitary. If we had only Γ part, the solution would in general decay, but not to zero if the projection of the initial data on the kernel of Γ is nonzero. Also, the rate of decay is independent of A of course. Now combine these two terms. When will this have a drastic effect of speeding up the decay and making it uniform over all

initial data? These questions have many applications to kinetic equations, such as Boltzmann and Fokker-Planck, and is very important in physics. Here is a sample of open problems.
1. Give conditions on L and Γ which makes $\|\psi^A\|$ go to zero faster with increase of A. A link with "hypocoercivity" theory, developed largely by Cedric Villani (this year's Fields winner) may prove fruitful here. I won't describe the details here, but this is a theory developed to address related questions (proving that certain semigroups are contracting).
2. Develop a quantitative version of Theorem 0.2. Namely, prove estimates on $A(\delta, \tau)$ appearing in Definition 0.1 depending on the properties of the flow. The same question makes sense in a more general abstract setting.
3. Prove an extension of Theorem 0.1 for manifolds which are not compact, like whole $\mathbb{R}^d$. Compactness really matters in the proof: it uses discreteness of the spectrum of Γ.

Working on any of these problems you will learn a lot of beautiful mathematics! Good progress on any of them will lead to a nice PhD.