Mathematical biology. This direction is related in spirit to the previous one, but mathematically it is quite different. The problem is motivated by the talk of Jeffrey Weiss (Colorado) which I attended at a multidisciplinary workshop at the IMA in March 2010. Prof. Weiss is an oceanographer, and he talked about coral spawning. Broadcast spawning is a fertilization strategy used by various benthic invertebrates (e.g. sea urchins, anemones, corals) whereby males and females release sperm and egg gametes into surrounding flow. The gametes are positively buoyant, and rise to the surface of the ocean. The sperm and egg are initially separated by the ambient water, and effective mixing is necessary for successful fertilization. The fertilized gametes form larva, which is negatively buoyant and tries to attach to the bottom of the ocean floor to start a new colony. For the coral spawning, field measurements of the fertilization rates are rarely below 5%, and are often as high as 90% - very efficient! On the other hand, numerical simulations based on heat equation with reaction models (called turbulent eddy diffusivity model, meaning to say the model accounts for the fluid flow motion by increasing diffusion in a heat equation and dropping the flow) predict fertilization rates of less than 1% due to strong dilution of gametes. To make it more clear, these simulations look at a system

$$\partial_t \rho_1 = \kappa_1 \Delta \rho_1 - M \rho_1 \rho_2$$
$$\partial_t \rho_1 = \kappa_2 \Delta \rho_1 - M \rho_1 \rho_2,$$

where $\rho_{1,2}$ are sperm and egg densities. The simulation looks how much of the original L^1 norm of the densities reacts and how much remains. According to the simulation, much more should remain than really does.

Prof. Weiss presented a more sophisticated model, taking into account instantaneous details of the flow not captured by effective diffusion approach. It is well known that the geometric structure of the fluid flow can be important for improving the reaction rate (this is related to part 2). The model of Prof. Weiss showed that vortex stirring can generally enhance the reaction rate, perhaps accounting for some of the discrepancy between simulation and experiment.

However, there is also experimental evidence that chemotaxis plays a role in coral and other marine animals fertilization: eggs release a chemical that attracts the sperm. Mathematically, chemotaxis has been extensively studied in the context of modeling mold and bacterial colonies. Since the original work of Keller-Segel where first PDE model of chemotaxis was introduced, there has been an enormous amount of effort devoted to studying the possible blow up and regularity of solutions, as well as asymptotic behavior and other properties. However, it seems that there has been no rigorous work on effects of chemotaxis for improved mixing and efficiency of biological reactions. This is a gap that should be filled - the topic is interesting both from mathematical and biological point of view.

Together with my collaborator Lenya Ryzhik, we already have some preliminary results. As the first step, we study the following model.

$$\partial_t \rho + (u \cdot \nabla)\rho - \chi \nabla(\rho \nabla (\Delta)^{-1} \rho) - \Delta \rho = -\rho^q, \;\; \rho(x,0) = \rho_0(x). \tag{1}$$

Here we have just one density, $\rho(x,t) \geq 0$, corresponding to the assumption that both male and female corals are located uniformly over all colony. In very basic approximation, we can then assume that the density of sperm and egg gametes is identical. The vector field u in (1) is divergence free, regular and passive, independent of ρ. This models the ambient ocean flow. The

third term is the standard chemotactic term, in the same form as it appears in the Keller-Segel equation. This term describes the tendency of $\rho(x,t)$ to move along the gradient of the chemical whose distribution is equal to $\Delta^{-1}\rho$. This is something you can see if you play with the formula a little - or ask me. The inverse Laplacian here comes from heat equation which describes propagation of the chemical generated by the density ρ. We also have diffusion and finally the reaction (fertilization) term $-\rho^q$. The value $q = 2$ is the most natural one, corresponding to the assumption that the fertilization rate is equal to the product of egg and sperm densities, but we generally consider $q \geq 2$. The relevant question then is the dynamics of the L^1 norm of the density $\rho(x,t)$ (which remains positive). The high efficiency fertilization corresponds to the L^1 norm of ρ becoming small with time, as almost all egg gametes are fertilized. We can prove the following

Theorem 0.1. *Assume u is smooth, bounded and divergence free, the initial data $\rho_0 \in S$ (Schwartz class), $d = 2$, $q > 2$, and chemotaxis is absent: $\chi = 0$. Then there exists a constant μ_0 depending only on d, q and $\rho_0(x)$ such that the L^1 norm of $\rho(x,t)$ remains greater than μ_0 for all times.*

Assume that $\chi > 0$. Then we have that $\|\rho(\cdot,t)\|_{L^1} \to c(q,\rho_0,\chi)$ as $t \to \infty$. Moreover, $c(q,\rho_0,\chi) \to 0$ as $\chi \to \infty$ (provided the initial data ρ_0 remains fixed).

In other words, the flow and diffusion alone can only go so far in enhancing reaction rate. There is a limit beyond which they are ineffective, no matter how strong is the flow or what structure it has. On the other hand, sufficiently strong chemotactic attraction can make fertilization as effective as needed. Thus, at least in the framework of model (1), chemotaxis is crucial for efficient fertilization!

We just started this project and Theorem 0.1 is, in some sense, picking low hanging fruit. First, there is an assumption that $q > 2$. The case $q = 2$ is more interesting, but also more difficult technically. Secondly, a more realistic model of the process would involve a system for two densities

$$\partial_t \rho_1 + (u \cdot \nabla)\rho_1 - \chi \nabla(\rho_1 \nabla(\Delta)^{-1}\rho_2) - \kappa\Delta\rho_1 = -(\rho_1\rho_2)^{q/2} \tag{2}$$

$$\partial_t \rho_2 + (u \cdot \nabla)\rho_2 - \kappa\Delta\rho_2 = -(\rho_1\rho_2)^{q/2} \tag{3}$$

$$\rho_1(x,0) = \rho_{1,0}(x), \;\; \rho_2(x,0) = \rho_{2,0}(x). \tag{4}$$

This is the system for which (with $\chi = 0$) numerical experiments of Prof. Weiss have been performed. Systems of PDE are always an order of magnitude more difficult for rigorous analysis than single equations, but I do think some interesting and non-trivial theorems can be proved here. Doing numerical simulations with chemotaxis present would also be extremely interesting.