GLOBAL WELL-POSEDNESS FOR A SLIGHTLY SUPERCRITICAL SURFACE QUASI-GEOSTROPHIC EQUATION

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Dedicated to Peter Constantin on the occasion of his 60th birthday.

ABSTRACT. We use a nonlocal maximum principle to prove the global existence of smooth solutions for a slightly supercritical surface quasi-geostrophic equation. By this we mean that the velocity field u is obtained from the active scalar θ by a Fourier multiplier with symbol $i\zeta^{\perp}|\zeta|^{-1}m(|\zeta|)$, where m is a smooth increasing function that grows slower than $\log\log|\zeta|$ as $|\zeta| \to \infty$.

1. INTRODUCTION

The surface quasi-geostrophic equation (SQG) has recently been a focus of research efforts by many mathematicians. It is probably the simplest physically motivated evolution equation of fluid mechanics for which, in the supercritical regime, it is not known whether solutions stay regular or if they can blow up in finite time. The equation is given by

$$\partial_t\theta + (u\cdot\nabla)\theta + \Lambda^{\alpha}\theta = 0, \quad \theta(\cdot,0) = \theta_0$$
$$u = \nabla^{\perp}\Lambda^{-1}\theta$$

on $(x,t) \in \mathbb{T}^2 \times [0,\infty)$, where $\Lambda = (-\Delta)^{1/2}$. The SQG equation appeared in the mathematical literature for the first time in [4], and since then has attracted significant attention, in part due to certain similarities with three dimensional Euler and Navier-Stokes equations. The equation has L^∞ maximum principle [12, 3], which makes the $\alpha = 1$ dissipation critical. It has been known for a while [12, 16] that the equation has global smooth solutions (for appropriate initial data) when $\alpha > 1$. The global regularity in the critical case has been settled independently by Kiselev-Nazarov-Volberg [11] (in the periodic setting) and Caffarelli-Vasseur [1] (in the whole space as well as in the local setting). A third proof of the same result was provided recently in [10]. All these proofs are quite different. The method of [1] is inspired by DeGiorgi iterative estimates, while the approach of [10] uses appropriate set of test functions and estimates on their evolution. The method of [11], on the other hand, is based on a new technique which can be called a nonlocal maximum principle. The idea is to prove that the evolution (1.1) preserves a certain modulus of continuity ω of the solution. The control is strong enough to give a uniform bound on $\|\nabla\theta\|_{L^\infty}$ in the critical case, which is sufficient for global regularity.

In the supercritical case, the only results available so far (for large initial data) have been on conditional regularity and finite time regularization of solutions. For instance, it was shown by Constantin and Wu [5] that if the solution is $C^{1-\alpha}$, then it is smooth. Finite time regularization has been proved by Silvestre [13] for α sufficiently close to 1, and for the whole dissipation range

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$0 < \alpha < 1$ by Dabkowski [6] (with an alternative proof of the latter result given in [9]). The issue of global regularity in the case $\alpha \in (0,1)$ remains an outstanding open problem.

Our goal here is to advance global regularity very slightly into the supercritical regime for the SQG equation. For technical reasons (and inspired by [2]), it is more convenient for us to introduce supercriticality in the velocity u rather than in the dissipation. Namely, let $m(\zeta) = m(|\zeta|)$ be a smooth, radial, non-decreasing function on $\mathbb{R}^2$, such that $m(\zeta) \geq 1$ for all $\zeta \in \mathbb{R}^2$. We shall consider the active scalar equation,

$$\partial_t \theta + (u \cdot \nabla)\theta + \Lambda\theta = 0, \quad \theta(\cdot, 0) = \theta_0 \tag{1.1}$$

$$u = \nabla^\perp \Lambda^{-1} m(\Lambda)\theta \tag{1.2}$$

on $(x,t) \in \mathbb{T}^2 \times [0,\infty)$, where $m(\Lambda)\theta$ is defined by its Fourier transform $(m(\Lambda)\theta)\widehat{\ }(\zeta) = m(\zeta)\widehat{\theta}(\zeta)$. Note that $m \equiv 1$ gives us the usual critical SQG equation. We shall consider symbols $m(\zeta)$ which for all sufficiently large $|\zeta|$ satisfy the growth condition

$$m(\zeta) \leq C\left(\log\log|\zeta|\right)^{1-\varepsilon} \tag{1.3}$$

for some $\varepsilon \in (0,1)$ and some $C > 0$. In addition we require that

$$\lim_{|\zeta|\to\infty} \frac{|\zeta| m'(\zeta)}{m(\zeta)} = 0 \tag{1.4}$$

and that the symbol m is of Hörmander-Mikhlin type, i.e., there exists $C > 0$ such that

$$|\zeta|^k |\partial_\zeta^k m(\zeta)| \leq C m(\zeta) \tag{1.5}$$

holds for all $\zeta \neq 0$, and all $k \in \{0, \ldots, d+2\}$. The main result of this paper is:

Theorem 1.1 (**Slightly supercritical SQG).** *Assume that $\theta_0 \in C^\infty(\mathbb{T}^2)$. If the symbol m satisfies* (1.3)–(1.5)*, then there exists a unique global C^∞ smooth solution θ of* (1.1)–(1.2).

Remark. The condition (1.4) can be improved, but is adapted here for the sake of simplicity. We can also carry the proof through with a growth condition a little weaker than (1.3), but the natural $m(\zeta)/\log\log|\zeta| \to 0$ as $|\zeta| \to \infty$ seems out of reach with our current technique.

The result we prove here is reminiscent of the slightly supercritical Navier-Stokes regularity result of Tao [15]. The challenge in the SQG case is that while regularity for critical Navier-Stokes is easy to prove by energy methods, there is no similarly simple proof of regularity for the critical SQG. The criticality of the SQG equation is controlled by the L^∞ norm, and the order of differentiation is the same in the nonlinearity and dissipation term. This makes global regularity for large data surprising at the first look. All three proofs of global regularity for critical SQG are somewhat subtle and involved. Scaling plays a crucial role in all existing proofs. The main contribution of this paper is to show that one can advance, at least a little, beyond the critical scaling.

To prove Theorem 1.1, we rely on the original method of [11]. This method is based on constructing a modulus of continuity $\omega(\xi)$, Lipshitz at zero and growing at infinity, which is respected by the critical SQG evolution: if the initial data θ_0 obeys ω, so does the solution $\theta(x,t)$ for every $t > 0$. By scaling, in the critical regime any rescaled modulus $\omega_B(\xi) = \omega(B\xi)$ is also preserved by the evolution. This allows, given smooth initial data θ_0, to find B such that θ_0 obeys ω_B and thus, due to preservation of ω_B, gain sufficient control of solution for all times. The unboundedness of ω is crucial for this argument; applying it with bounded ω would correspond to controlling only initial data of limited size. It appears that the maximal growth of ω one can afford in the critical SQG case is a double logarithm, dictated by balance of nonlinear and dissipative term estimates. The idea of the proof of Theorem 1.1, and the key observation of this paper, is that it is possible to trade some of this growth in ω for a slightly rougher velocity u (or, likely, slightly weaker dissipation). In the process, one loses critical scaling, but the argument can be made to work by manufacturing a family of moduli ω_B preserved by the evolution which are no longer a single rescaled modulus.

We anticipate that the approach we develop here will have other applications. In particular, it can be applied to a slightly supercritical Burgers equation. In this case, one can prove global regularity for a more singular equation, supercritical by almost a logarithmic multiplier. This is due to the existence of moduli with logarithmic growth conserved by the evolution. Consideration of the Burgers equation, as well as applications to modified SQG, and the case of supercritical dissipation is postponed to a subsequent publication [7].

2. Preliminaries

The local and conditional regularity for the SQG-type equations is by now standard. In particular, we have

Proposition 2.1 (Local existence of smooth solution). *Given $\theta_0 \in H^s(\mathbb{T}^2)$, for some $s > 1$, there exists $T > 0$ and a solution $\theta(\cdot, t) \in C([0,T], H^s) \cap C^\infty((0,T] \times \mathbb{T}^2)$ of (1.1)–(1.2). Moreover, the solution may be continued as a smooth solution beyond T as long as $\|\nabla\theta\|_{L^1(0,T;L^\infty(\mathbb{T}^2))} < \infty$.*

The proof of a similar result with standard SQG velocity and critical or supercritical dissipation can be found, for example, in [8]. Modifying the constitutive law for u by the multiplier m does not create any essential additional difficulties in the argument.

Definition 2.2 (Modulus of continuity). *We call a function $\omega\colon (0,\infty) \to (0,\infty)$ a modulus of continuity if ω is increasing, continuous, concave, piecewise C^2 with one sided derivatives, and it additionally satisfies $\omega'(0+) = \infty$ or $\omega''(0+) = -\infty$. We say that a smooth function f* obeys *the modulus of continuity ω if $|f(x) - f(y)| < \omega(|x-y|)$ for all $x \neq y$.*

We recall that if $f \in C^\infty(\mathbb{T}^2)$ obeys the modulus ω, then $\|\nabla f\|_{L^\infty} < \omega'(0)$. In addition, observe that a function $f \in C^\infty(\mathbb{T}^2)$ automatically obeys any modulus of continuity $\omega(\xi)$ that lies above the function $\min\{\xi\|\nabla f\|_{L^\infty}, 2\|f\|_{L^\infty}\}$.

We will construct a family of moduli of continuity ω_B that will be preserved by the evolution. To prove this nonlocal maximum principle, we will use the following outline. The proofs of Lemmas 2.3 and 2.5 below can be found in [11].

Lemma 2.3 (Breakthrough scenario). *Assume ω is a modulus of continuity such that $\omega(0+) = 0$ and $\omega''(0+) = -\infty$. Suppose that the initial data θ_0 obeys ω. If the solution $\theta(x,t)$ violates ω at some positive time, then there must exist $t_1 > 0$ and $x \neq y \in \mathbb{T}^2$ such that*

$$\theta(x,t_1) - \theta(y,t_1) = \omega(|x-y|),$$

and $\theta(x,t)$ obeys ω for every $0 \le t < t_1$.

Let us consider the breakthrough scenario for a modulus ω. A simple computation shows that

$$\begin{aligned}\partial_t\left(\theta(x,t) - \theta(y,t)\right)|_{t=t_1} &= u\cdot\nabla\theta(y,t_1) - u\cdot\nabla\theta(x,t_1) + \Lambda\theta(y,t_1) - \Lambda\theta(x,t_1)\\ &\le |u(x,t_1) - u(y,t_1)|\omega'(\xi) + \Lambda\theta(y,t_1) - \Lambda\theta(x,t_1). \qquad (2.1)\end{aligned}$$

If we can show that the expression in (2.1) must be strictly negative, we obtain a contradiction: ω cannot be broken, and hence it is preserved. To estimate (2.1) we need

Lemma 2.4 (**Modulus of continuity for the drift velocity).** *Assume that θ obeys the modulus of continuity ω, and that the drift velocity is given as $u = \nabla^\perp \Lambda^{-1} m(\Lambda)\theta$. Then u obeys the modulus of continuity Ω defined as*

$$\Omega(\xi) = A\left(\int_0^\xi \frac{\omega(\eta)m(\eta^{-1})}{\eta}d\eta + \xi\int_\xi^\infty \frac{\omega(\eta)m(\eta^{-1})}{\eta^2}d\eta\right) \tag{2.2}$$

for some positive constant$A \geq 1$ that only depends on the function m.

The proof of Lemma 2.4 shall be given in the Appendix. For the dissipative terms, we have:

Lemma 2.5 (**Dissipation control).** *Assume we are in a breakthrough scenario as in Lemma 2.3. Then*

$$\begin{aligned}\Lambda\theta(y,t_1) - \Lambda\theta(x,t_1) \leq \mathcal{D}(\xi) \equiv \frac{1}{\pi}\int_0^{\xi/2} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}d\eta \\ + \frac{1}{\pi}\int_{\xi/2}^\infty \frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^2}d\eta.\end{aligned} \tag{2.3}$$

Given the three Lemmas above and (2.1), in order to verify the preservation of ω for all time, it is sufficient to check that $\Omega(\xi)\omega'(\xi) + \mathcal{D}(\xi) < 0$ for every $\xi > 0$.

3. Proof of Main Theorem

The main difference between our argument here and [11] is that since (1.1)–(1.2) is beyond the critical scaling, one cannot use $\omega_B(\xi) = \omega(B\xi)$ to construct the needed family of moduli of continuity, from a fixed modulus ω.

3.1. A suitable family of moduli of continuity.

For $B \geq 1$, we shall consider the modulus of continuity ω_B defined as the continuous function with

$$\omega_B(\xi) = B\xi - (B\xi)^{1+\varepsilon/8}, \text{ for all } 0 < \xi \leq \delta(B) \tag{3.1}$$

$$\omega_B'(\xi) = \frac{\gamma}{\xi(1+\ln(\xi/\delta(B)))m(B)^{1+\varepsilon/2}}, \text{ for all } \xi > \delta(B) \tag{3.2}$$

where we let

$$\delta(B) = \frac{\kappa}{Bm(B)^{1+\varepsilon/4}} \tag{3.3}$$

and κ, γ are sufficiently small constants depending only on m, ε and A from (2.2), to be chosen later. To verify that ω_B is a modulus of continuity, it is clear that concavity is the only nontrivial aspect to check. To address the latter note that

$$\begin{aligned}\omega_B'(\delta(B)-) = B - \left(1+\frac{\varepsilon}{8}\right)B^{1+\varepsilon/8}\delta(B)^{\varepsilon/8} \geq B\left(1 - 2\kappa^{\varepsilon/8}\frac{1}{m(B)^{\varepsilon(4+\varepsilon)/32}}\right) \\ \geq B\left(1-2\kappa^{\varepsilon/8}\right) > \frac{B}{2}\end{aligned} \tag{3.4}$$

if κ is sufficiently small. On the other hand we have

$$\omega_B'(\delta(B)+) = \frac{\gamma}{\delta(B)m(B)^{1+\varepsilon/2}} = \frac{2\gamma}{\kappa m(B)^{\varepsilon/4}}\cdot\frac{B}{2} < \frac{B}{2} \tag{3.5}$$

for all $\gamma < \kappa/2$, since $m(B) \geq 1$. Together, (3.4) and (3.5) show that ω_B is concave.

Let us denote $\Omega_B(\xi)$ and $\mathcal{D}_B(\xi)$ respectively the modulus of the velocity u given by (2.2) and dissipation estimate (2.3) corresponding to $\omega_B(\xi)$.

It is sufficient to prove two things: that each initial data θ_0 obeys some modulus of continuity ω_B for a suitable $B \geq 1$, and that the expression in (2.1) when computed for each ω_B is strictly negative for all $\xi > 0$.

3.2. **Modulus of continuity for the initial data.** First we show that any initial data $\theta_0 \in C^\infty(\mathbb{T}^2)$ obeys a modulus of continuity ω_B for some sufficiently large B. As noted earlier, this is achieved if we find a sufficiently large B such that $\omega_B(\xi) > \min\{\xi\|\nabla\theta_0\|_{L^\infty}, 2\|\theta_0\|_{L^\infty}\}$ for all $\xi > 0$. Observe that due to concavity of ω_B it is sufficient to find B such that

$$\omega_B\left(\frac{2\|\theta_0\|_{L^\infty}}{\|\nabla\theta_0\|_{L^\infty}}\right) \geq 2\|\theta_0\|_{L^\infty}.$$

However, note that for every fixed $a > 0$, we have $a > \delta(B)$ if B is sufficiently large, and

$$\int_{\delta(B)}^{a} \frac{\gamma}{\xi(1+\log(\xi/\delta(B)))m(B)^{1+\varepsilon/2}} d\xi = \frac{\gamma}{m(B)^{1+\varepsilon/2}} \log(1+\log(a/\delta(B))) \to \infty$$

as $B \to \infty$ due to our assumption (1.3) on growth of m. This shows that any smooth θ_0 obeys a modulus of continuity ω_B if B is chosen large enough.

3.3. **Evolution of the modulus of continuity.** We shall now prove that if κ is chosen sufficiently small (depending only on $\varepsilon, m,$ and A), and γ is chosen sufficiently small (depending only on $\kappa, \varepsilon, m,$ and A), then the expression (2.1) is strictly negative, i.e. $\Omega_B(\xi)\omega_B'(\xi) + \mathcal{D}_B(\xi) < 0$, for all $\xi > 0$. Note that neither κ, nor γ will depend on $B \geq 1$.

The case $0 < \xi \leq \delta(B)$**.** We first observe that $\omega_B'(\xi) \leq B$ for all $\xi \in (0, \delta(B)]$. Using concavity of ω and the mean value theorem, we can estimate

$$\mathcal{D}_B(\xi) \leq \frac{1}{\pi}\xi\omega_B''(\xi) = -\left(1+\frac{\varepsilon}{8}\right)\frac{\varepsilon}{8}B^{1+\varepsilon/8}\xi^{\varepsilon/8} \leq -\frac{\varepsilon}{8}B^{1+\varepsilon/8}\xi^{\varepsilon/8}. \tag{3.6}$$

The main issue is to estimate the contribution from $\Omega_B(\xi)$. From (2.2) we have that

$$\begin{aligned}\Omega_B(\xi)\omega_B'(\xi) &\leq AB\left(B\int_0^\xi m(\eta^{-1})d\eta + B\xi\int_\xi^{\delta(B)} \frac{m(\eta^{-1})}{\eta}d\eta + \xi\int_{\delta(B)}^\infty \frac{\omega_B(\eta)m(\eta^{-1})}{\eta^2}d\eta\right)\\ &\leq AB\left(2B\xi m(\xi^{-1}) + B\xi m(\xi^{-1})\ln\frac{\delta(B)}{\xi} + \xi m(\xi^{-1})\int_{\delta(B)}^\infty \frac{\omega_B(\eta)}{\eta^2}d\eta\right) \end{aligned}\tag{3.7}$$

for some sufficiently large constant C, depending only on the function m. In the second inequality of (3.7) we have used the monotonicity of m and the inequality

$$\int_0^\xi m(\eta^{-1})d\eta \leq 2\xi m(\xi^{-1}) \tag{3.8}$$

which holds for all sufficiently small ξ due to (1.4). Indeed, since $\xi \leq \delta(B) < \kappa$, by letting κ be sufficiently small, we have from (1.4) that $2rm'(r) \leq m(r)$ holds for all $r = \eta^{-1} \geq \kappa^{-1}$. The latter inequality gives that the function $m(r)r^{-1/2}$ is non-increasing for $r \geq \kappa^{-1}$. Then (3.8) follows from

$$m(\eta^{-1}) \leq \xi^{1/2}m(\xi^{-1})\eta^{-1/2} \tag{3.9}$$

and integration from 0 to ξ.

In order to estimate $\int_{\delta(B)}^{\infty} \omega_B(\eta)/\eta^2 d\eta$, we integrate by parts and use the slow growth of ω_B (cf. (1.3)) to obtain

$$\int_{\delta(B)}^{\infty} \frac{\omega_B(\eta)}{\eta^2} d\eta \leq \frac{\omega_B(\delta(B))}{\delta(B)} + \int_{\delta(B)}^{\infty} \frac{\gamma}{\eta^2(1+\ln(\eta/\delta(B)))m(B)^{1+\varepsilon/2}} d\eta$$
$$\leq B + \frac{\gamma}{\delta(B)m(B)^{1+\varepsilon/2}} = B + \frac{\gamma B}{\kappa m(B)^{\varepsilon/4}} \leq 2B \tag{3.10}$$

if $\gamma < \kappa$, since $m(B) \geq 1$. Note that to avoid a circular argument, κ will be chosen below independently of γ. Combining (3.6) with (3.7) and (3.10) we obtain

$$\Omega_B(\xi)\omega_B'(\xi) + \mathcal{D}_B(\xi) \leq AB^2\xi m(\xi^{-1}) \left(4 + \ln \frac{\delta(B)}{\xi}\right) - \frac{\varepsilon}{8} B^{1+\varepsilon/8}\xi^{\varepsilon/8}. \tag{3.11}$$

Finally, we choose κ so that the right side of (3.11) is strictly negative for all $\xi \in (0, \delta(B)]$. Set $\rho = B\xi$.

Lemma 3.1 (**Choosing** κ)**.** *There exists $\kappa = \kappa(\varepsilon, m, A)$, independent of B, such that if $0 < \rho \leq \kappa m(B)^{-1-\varepsilon/4}$, then*

$$A\rho^{1-\varepsilon/8} m(B\rho^{-1}) \left(4 + \ln \frac{\kappa}{\rho m(B)^{1+\varepsilon/4}}\right) < \frac{\varepsilon}{8} \tag{3.12}$$

for all $B \geq 1$.

Proof of Lemma 3.1. First, fix some $\sigma \in (0, \varepsilon/14)$. Due to (1.4), we have that $rm'(r) \leq \sigma m(r)$ for all $r \geq \kappa^{-1}$ provided that κ is sufficiently small, and hence in this range the function $m(r)r^{-\sigma}$ is non-increasing. Therefore, $m(B\rho^{-1}) \leq \rho^{-\sigma} m(B)$, and the left side of (3.12) may be bounded from above by

$$A\rho^{1-\varepsilon/8-\sigma} m(B) \left(4 + \ln \frac{1}{\rho m(B)^{1+\varepsilon/4}}\right) \leq A \left(\rho m(B)^{1+\varepsilon/4}\right)^{4/(\varepsilon+4)} \left(4 + \ln \frac{1}{\rho m(B)^{1+\varepsilon/4}}\right) \tag{3.13}$$

as long as $1 - \varepsilon/8 - \sigma \geq 4/(\varepsilon + 4)$. The latter holds for $\varepsilon \in (0, 1)$ if $\sigma < \varepsilon/14$. Since $\lim_{r\to 0+} r^{4/(\varepsilon+4)}(4 + \ln 1/r) = 0$, there exists a sufficiently small κ, depending only on ε, and A, such that $r^{4/(\varepsilon+4)}(4 + \ln 1/r) \leq \varepsilon/(8A)$ for all $r \leq \kappa$. Letting $r = \rho m(B)^{1+\varepsilon/4}$ concludes the proof of the lemma. □

We choose κ small enough so that Lemma 3.1 is satisfied and also such that $\kappa < 2^{-8/\varepsilon}$, a condition that we will need later.

The case $\xi > \delta(B)$**.** We observe that for each $B \geq 1$ the modulus of continuity ω_B satisfies

$$\omega_B(2\xi) \leq \frac{3}{2}\omega_B(\xi), \text{ for all } \xi \geq \delta(B). \tag{3.14}$$

Indeed due to the definition (3.2) of ω_B, we have

$$\omega_B(2\xi) \leq \omega_B(\xi) + \frac{2\gamma}{m(B)^{1+\frac{\varepsilon}{2}}}$$

for every $\xi \geq \delta(B)$. But $\omega_B(\xi) \geq \omega(\delta(B))$. A simple calculation shows that taking

$$\gamma < \frac{1}{4}(\kappa - \kappa^{1+\frac{\varepsilon}{8}})$$

is sufficient for (3.14) to hold. Using (3.14), we estimate

$$\mathcal{D}_B(\xi) \leq \frac{1}{\pi}\int_{\xi/2}^{\infty} \frac{\omega_B(2\eta+\xi)-\omega_B(2\eta-\xi)-\omega_B(2\xi)-\frac{1}{2}\omega_B(\xi)}{\eta^2}d\eta \leq -\frac{1}{2\pi}\frac{\omega_B(\xi)}{\xi} \tag{3.15}$$

holds for all $\xi > \delta(B)$. Next, let us estimate the term arising from $\Omega_B(\xi)\omega_B'(\xi)$ in (2.1), namely

$$A\omega_B'(\xi)\left(\int_0^{\xi}\frac{\omega_B(\eta)m(\eta^{-1})}{\eta}d\eta + \xi\int_{\xi}^{\infty}\frac{\omega_B(\eta)m(\eta^{-1})}{\eta^2}d\eta\right). \tag{3.16}$$

We first bound

$$\begin{aligned}\int_0^{\xi}\frac{\omega_B(\eta)m(\eta^{-1})}{\eta}d\eta &\leq B\int_0^{\delta(B)} m(\eta^{-1})d\eta + \int_{\delta(B)}^{\xi}\frac{\omega_B(\eta)m(\eta^{-1})}{\eta}d\eta\\ &\leq 2B\delta(B)m(\delta(B)^{-1}) + \omega_B(\xi)m(\delta(B)^{-1})\ln\frac{\xi}{\delta(B)} \end{aligned}\tag{3.17}$$

for all $\xi > \delta(B)$. Using (1.4), we estimate $m(\delta(B)^{-1}) \leq \kappa^{-\sigma}m(B)^{1+\sigma(1+\varepsilon/4)}$, where we recall $0<\sigma<\varepsilon/14$, and hence the right hand side of (3.17) is bounded from above by

$$\frac{2\kappa^{1-\sigma}}{m(B)^{\varepsilon/4-\sigma(1+\varepsilon/4)}} + \omega_B(\xi)\frac{m(B)^{1+\sigma(1+\varepsilon/4)}}{\kappa^{\sigma}}\ln\frac{\xi}{\delta(B)}. \tag{3.18}$$

Furthermore, we have

$$\begin{aligned}\xi\int_{\xi}^{\infty}\frac{\omega_B(\eta)m(\eta^{-1})}{\eta^2}d\eta &\leq \xi m(\xi^{-1})\int_{\xi}^{\infty}\frac{\omega_B(\eta)}{\eta^2}d\eta\\ &= \xi m(\delta(B)^{-1})\left(\frac{\omega_B(\xi)}{\xi} + \frac{\gamma}{m(B)^{1+\varepsilon/2}}\int_{\xi}^{\infty}\frac{1}{\eta^2(1+\ln(\eta/\delta(B)))}d\eta\right)\\ &\leq \frac{m(B)^{1+\sigma(1+\varepsilon/4)}}{\kappa^{\sigma}}\left(\omega_B(\xi) + \frac{\gamma}{m(B)^{1+\varepsilon/2}}\right). \end{aligned}\tag{3.19}$$

Therefore, inserting the bounds (3.18) and (3.19) into (3.16), and using $\kappa \leq 1 \leq m(B)$, we obtain

$$\begin{aligned}\Omega_B(\xi)\omega_B'(\xi) &\leq \frac{A\gamma}{\kappa^{\sigma}m(B)^{\varepsilon/2-\sigma(1+\varepsilon/4)}}\frac{\omega_B(\xi)}{\xi} + \frac{A(\gamma+2)\gamma}{\kappa^{\sigma}m(B)^{\varepsilon/2+(1-\sigma)(1+\varepsilon/4)}}\frac{1}{\xi}\\ &\leq \frac{A\gamma}{\kappa^{\sigma}}\frac{\omega_B(\xi)}{\xi} + \frac{3A\gamma}{\kappa^{\sigma}m(B)^{1+\varepsilon/2}}\frac{1}{\xi} \end{aligned}\tag{3.20}$$

since $\sigma < \varepsilon/14$. The negativity of $\Omega_B(\xi)\omega_B'(\xi) + \mathcal{D}_B(\xi)$ follows from the below Lemma and estimates (3.15), (3.20), thereby concluding the estimates for the case $\xi > \delta(B)$ and proof of Theorem 1.1.

Lemma 3.2 (Choosing γ). *There exists a $\gamma > 0$, depending only on A, σ, and κ, such that*

$$-\frac{1}{2\pi}\frac{\omega_B(\xi)}{\xi} + \frac{A\gamma}{\kappa^{\sigma}}\frac{\omega_B(\xi)}{\xi} + \frac{3A\gamma}{\kappa^{\sigma}m(B)^{1+\varepsilon/2}}\frac{1}{\xi} < 0 \tag{3.21}$$

holds for all $\xi > \delta(B)$ and all $B \geq 1$.

Proof of Lemma 3.2. If we let $\gamma \leq 1$ be small enough so that $\gamma A/\kappa^{\sigma} < 1/(4\pi)$, and use the fact that ω_B is increasing, we obtain that the expression on the left hand side of (3.21) is bounded by

$$-\frac{1}{4\pi}\frac{\omega_B(\xi)}{\xi} + \frac{3A\gamma}{\kappa^{\sigma}m(B)^{1+\varepsilon/2}}\frac{1}{\xi} \leq \frac{1}{\xi m(B)^{1+\varepsilon/2}}\left(\frac{3A\gamma}{\kappa^{\sigma}} - \frac{m(B)^{1+\varepsilon/2}\omega_B(\delta(B))}{4\pi}\right). \tag{3.22}$$

Since, whenever $\kappa \leq 2^{-8/\varepsilon}$, we have

$$m(B)^{1+\varepsilon/2}\omega_B(\delta(B)) \geq \kappa - \frac{\kappa^{1+\varepsilon/8}}{m(B)^{\varepsilon(4+\varepsilon)/32}} \geq \frac{\kappa}{2} \tag{3.23}$$

it is enough to let γ be such that

$$\gamma\frac{3A}{\kappa^{\sigma}} < \frac{\kappa}{8\pi} \tag{3.24}$$

in order to guarantee that the right side of (3.22) is strictly negative, concluding the proof of the lemma. □

4. Appendix

Here we give details regarding the proof of Lemma 2.4. Let $m(\zeta)$ be a continuous, radial, non-decreasing function on $\mathbb{R}^d$, smooth on $\mathbb{R}^d$, with $m(\zeta) = m(|\zeta|) \geq 1$ for all $\zeta \in \mathbb{R}^d$. Assume that $m(\zeta)$ satisfies the Hörmander-Mikhlin-type condition (cf. [14])

$$|\zeta|^k|\partial_\zeta^k m(\zeta)| \leq Cm(\zeta) \tag{4.1}$$

for some $C \geq 1$, all $\zeta \neq 0$, and all $k \in \{0, \dots, d+2\}$. In addition we require that

$$\lim_{|\zeta|\to\infty} \frac{|\zeta|m'(\zeta)}{m(\zeta)} = 0. \tag{4.2}$$

The following lemma gives estimates on the distribution K whose Fourier transform is $i\zeta_j|\zeta|^{-1}m(\zeta)$, for any $j \in \{1, \dots, d\}$.

Lemma 4.1 (Kernel estimate). *Let $K(x)$ be the kernel of the operator $\partial_j\Lambda^{-1}m(\Lambda)$, where m is smooth on $\mathbb{R}^d$, radial, non-decreasing in radial variable, and satisfies the conditions* (4.1)–(4.2). *Then we have*

$$|K(x)| \leq C|x|^{-d}m(|x|^{-1}) \tag{4.3}$$

and

$$|\nabla K(x)| \leq C|x|^{-d-1}m(|x|^{-1}) \tag{4.4}$$

for all $x \neq 0 \in \mathbb{R}^d$.

Proof of Lemma 4.1. Consider a smooth non-increasing radial cutoff function $\eta(\zeta) = \eta(|\zeta|)$ which is identically 1 on $|\zeta| \leq 1/2$, and vanishes identically on $|\zeta| \geq 1$. For $R > 0$, let $\eta_R(|\zeta|) = \eta(|\zeta|/R)$. Then, for $R > 0$ to be chosen later, we decompose

$$K(x) = C\int_{\mathbb{R}^d}\eta_R(\zeta)m(\zeta)i\zeta_j|\zeta|^{-1}e^{i\zeta\cdot x}d\zeta + C\int_{\mathbb{R}^d}(1-\eta_R(\zeta))m(\zeta)i\zeta_j|\zeta|^{-1}e^{i\zeta\cdot x}d\zeta = K_1(x) + K_2(x).$$

Since $m(\zeta)$ is increasing, and η_R is supported on B_R, we may bound $|K_1(x)| \leq CR^d m(R)$. On the other hand, upon integrating by parts $d+2$ times, using (4.1) and the fact that $\partial_\zeta(1-\eta_R(\zeta))$ is supported on $R/2 \leq |\xi| \leq R$, we obtain

$$\begin{aligned}|K_2(x)| &\leq C|x|^{-d-2}\int_{\mathbb{R}^d}\left|\partial_\zeta^{d+2}\left((1-\eta_R)(\zeta)m(\zeta)i\zeta_j|\zeta|^{-1}\right)\right|d\zeta\\ &\leq C|x|^{-d-2}\left(R^{-d-2}\int_{R/2\leq|\zeta|\leq R}m(\zeta)d\zeta + \int_{|\zeta|\geq R/2}|\zeta|^{-d-2}m(\zeta)d\zeta\right).\end{aligned} \tag{4.5}$$

Observe that condition (4.2) shows there exists some $r > 0$ such that for all $|\zeta| \geq r$ we have $2|\zeta|m'(\zeta) \leq m(\zeta)$, and hence the function $|\zeta|^{-1/2}m(|\zeta|)$ is non-increasing for $|\zeta| \geq r$. Consider first small x, $|x| \leq 1/2r$. Letting $R = |x|^{-1}$, we have that $R/2 \geq r$. Using the facts that $m(|\zeta|)$ is non-decreasing, and $|\zeta|^{-1/2}m(|\zeta|)$ is non-increasing on $|\zeta| \geq r$, we obtain

$$|K_2(x)| \leq C|x|^{-d}m(|x|^{-1}) \tag{4.6}$$

which upon recalling the earlier bound on K_1 concludes the proof of (4.3) for small x. For $|x| \geq 1/2r$, we can set $R = 1$ and obtain that

$$|K_2(x)| \leq C|x|^{-d-2},$$

since due to (4.2) and the continuity of m we have $|m(\zeta)| \leq C(m)|\zeta|^{1/2}$. On the other hand,

$$K_1(x) = C\int_{\mathbb{R}^d}(c_0 i\zeta_j|\zeta|^{-1} + \varphi(\zeta))e^{i\zeta\cdot x}\,d\zeta,$$

where c_0 is a constant and $\varphi(\zeta) \in C_0^\infty$. This gives the bound

$$|K(x)| \leq C|x|^{-d},$$

which together with (4.6) implies (4.3) for $|x| \geq 1/2r$. The bounds for $\nabla K(x)$ are obtained in the same fashion, the only difference being an extra factor of ζ in the estimates. $\square$

Having estimated the kernel of the operator $\theta \mapsto u$, we are now ready to estimate the modulus of continuity of the velocity u, in terms of the modulus of continuity of the active scalar θ.

Proof of Lemma 2.4. The proof is similar to that of [11, Lemma]. Fix $x \neq y$, and let $\xi = |x-y|$. Since $u = \nabla^\perp(\Lambda^{-1}m(\Lambda)\theta)$ we have that $\int_{|x|=1}K(x)d\sigma(x) = 0$, and hence we may bound

$$\begin{aligned}u(x) - u(y) = &\int_{|x-z|\leq 2\xi}K(x-z)(\theta(z)-\theta(x))dz - \int_{|y-z|\leq 2\xi}K(y-z)(\theta(z)-\theta(y))dz\\ &+ \int_{|x-z|\geq 2\xi}K(x-z)(\theta(z)-\theta(\bar{z}))dz - \int_{|y-z|\geq 2\xi}K(y-z)(\theta(z)-\theta(\bar{z}))dz\end{aligned}$$

where the integrals are taken in the principal value sense, and $\bar{z} = (x+y)/2$. Using the estimates on the kernel K from Lemma 4.1, we obtain

$$\begin{aligned}|u(x)-u(y)| \leq C&\int_0^{2\xi}\frac{m(\eta^{-1})\omega(\eta)}{\eta}d\eta + \int_{|\bar{z}-z|\geq 3\xi}|K(x-z)-K(y-z)||\theta(z)-\theta(\bar{z})|dz\\ &+ \int_{3\xi/2\leq|\bar{z}-z|\leq 3\xi}(|K(x-z)|+|K(y-z)|)\,|\theta(z)-\theta(\bar{z})|dz.\end{aligned} \tag{4.7}$$

To estimate the second integral on the right hand side, note that for $|z-\bar{z}| \geq 3\xi$, by the mean value theorem and (4.4), we have

$$|K(x-z)-K(y-z)| \leq C\xi|z-\bar{z}|^{-3}m(|z-\bar{z}|^{-1}).$$

Here we use that $m(sr) \leq s^C m(r)$ holds by (4.1) for $s > 1$. The third integral on the right hand side of (4.7) is bounded using (4.3) and we obtain

$$|u(x)-u(y)| \leq C\int_0^{3\xi}\frac{m(\eta^{-1})\omega(\eta)}{\eta}d\eta + C\xi\int_{3\xi}^\infty\frac{m(\eta^{-1})\omega(\eta)}{\eta^2}d\eta \tag{4.8}$$

for all $\xi \neq 0$. The final result then follows from (4.8) using the concavity of ω and the monotonicity of m. $\square$

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