# Transfer matrices and transport for Schrödinger operators

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**Abstract**

We provide a general lower bound on the dynamics of one dimensional Schrödinger operators in terms of transfer matrices. In particular it yields a non trivial lower bound on the transport exponents as soon as the norm of transfer matrices does not grow faster than polynomially on a set of full Lebesgue measure, and regardless of the nature of the spectrum. We also develop some general analysis of wave-packets that enables one to characterize transports exponents at low and large moments. As an application of our general lower bound, we study a Schrödinger operator with random decaying potential, providing a new example of Schrödinger operators with point spectrum and nontrivial quantum transport. We also investigate sparse potential, as well as we revisit almost Mathieu as given by the celebrated pathological example of Del Rio, Jitomirskaya, Last.

## Contents

# 1 Introduction

We consider discrete, resp. continuous, Schrödinger operators $H = -\Delta + V$, with Dirichlet boundary conditions, acting on $\mathcal{H} = \ell^2([1,+\infty))$, resp. $\mathcal{H} = \mathrm{L}^2([0,+\infty))$. On the lattice the Laplacian $-\Delta$ is the finite differences operator $(-\Delta\psi)(n) = \psi(n+1) + \psi(n-1)$, and in the continuum $-\Delta\psi = -\psi''$. The goal of the present paper is to study the dynamics associated to $H$ as it is determined by the Schrödinger equation. We provide lower bounds on the dynamics that involve the behavior of the transfer matrices $T(E,N,0)$. Our main general result relates the amplitude of $T(E,N,0)$ to the evaluated dynamics at time $T \sim N$. Although our main lower bound and its consequences do not depend on the nature of the spectrum of $H$ (and we would rather consider this as an advantage than as a shortcoming), a typical range of applications will be the presence of singular spectrum (pure point or singular continuous).

To study the evolution of the dynamics, we define the averaged moments of order $p$ associated to the initial state localized at the origin and with energy "localized" in an *open* interval $I$, and at time T, by

$$\mathbb{M}(p,f,T) \quad = \quad \frac{2}{T}\int_0^\infty \mathrm{e}^{-2t/T}\left\| \langle X\rangle^{p/2}\mathrm{e}^{-iHt}f(H)\psi_0\right\|_{\mathcal{H}}^2 \mathrm{d}t, \tag{1.1}$$

where $f \in \mathcal{C}^\infty_{c,+}(I)$, an infinitely differentiable positive function with compact support in $I$, and

$$\begin{cases} \psi_0 = \delta_1 & \text{on } \mathcal{H} = \ell^2([1,+\infty)) \\ \psi_0 = \chi_0 & \text{on } \mathcal{H} = \mathrm{L}^2([0,+\infty)), \end{cases} \tag{1.2}$$

where $\chi_0$ is the characteristic function of the unit cube centered at 0 (we could also consider initial states $\psi_0 \in \mathrm{L}^2([0,1])$, i.e. such that $\psi_0 = \chi_0\psi_0$), and $\delta_1$ is the element of $\ell^2([1,+\infty))$ equal to 1 at $x = 1$ and zero everywhere else. We shall denote by $\mathcal{C}^\infty_{c,+}(I)$ the set of smooth functions compactly supported on $I$ and taking nonnegative values. To investigate the polynomial behavior in $T$ of $\mathbb{M}(p,f,T)$ we define the lower and upper transport exponents corresponding to a function $f \in \mathcal{C}^\infty_{c,+}$,

$$\beta^-(p,f) = \liminf_{T\to\infty}\frac{\log \mathbb{M}(p,f,T)}{p\log T}, \quad \beta^+(p,f) = \limsup_{T\to\infty}\frac{\log \mathbb{M}(p,f,T)}{p\log T}. \tag{1.3}$$

In the recent years, the propagation rates of wavepackets, and in particular behavior of the moments of initially localized states has been an object of active research; see for example [G, C, La, SBB, BCM, BGT1, BGT2, BSB, DR+1, DR+2, GSB1, GSB2, BGSB, JSBS, KL, DT, Tc1]. First works on the subject focused on the relation between regularity of the spectral measure (usually expressed in terms related to the Hausdorff dimension) and dynamics. Guarneri [G] proved that if the spectral measure is uniformly $\alpha$-continuous, then (in our notation) $\beta^-(p,f) \geq \alpha/d$ for any $f \in C_0^\infty(\mathbb{R})$ (where $d$ is the dimension of the coordinate space - here $d = 1$). His results were extended by Combes [C], Last [La] and Guarneri, Schulz-Baldes [GSB1]. Motivated by numerical works of Mantica [Ma], a new approach using generalized fractal dimensions has been developed by Barbaroux and two of us in [BGT1, BGT2, BGT3]. We also refer to Guarneri and Schulz-Baldes [GSB2] where similar ideas are developed under more restrictive hypotheses. This approach provides a lower bound of the moments $\mathbb{M}(p, f, T)$ in terms of integrals that we would like to call "transport integrals":

$$I_\mu(q, \varepsilon) = \int_{\mathrm{supp}\mu} \mu(x - \varepsilon, x + \varepsilon)^{q-1} \, \mathrm{d}\mu(x) \,, \tag{1.4}$$

where $q \in (0, 1), \varepsilon > 0$ and $\mu = \mu_{f(H)\psi_0}$ is the spectral measure associated to the initial state $f(H)\psi_0$. The estimate that is proved reads

$$\mathbb{M}(p, f, T) \geq \left( \frac{C}{\log T} \, I_\mu(q, T^{-1}) \right)^{\frac{1}{q}} \,, \qquad q = \frac{1}{1 + p/d} \,, \tag{1.5}$$

for some constant $C > 0$, and with $\mu = \mu_{f(H)\psi_0}$ and $d$ the dimension of the physical space. A key point in the analysis of the present paper will be the equivalence,

$$I_\mu(q, \varepsilon) \sim \frac{1}{\varepsilon} \int_{\mathbb{R}} \mu(x - \varepsilon, x + \varepsilon)^q \, \mathrm{d}x \,, \quad q > 0, \tag{1.6}$$

proved in [BGT3] for transport integrals; where by $f \sim g$ we mean the existence of a universal constant $c$ such that $c^{-1} f \leq g \leq cf$. Following (1.5), [BGT1, BGT2]'s result then reads, for compactly supported functions $f$,

$$\beta^\pm(p, f) \geq \frac{1}{d} D_\mu^\pm \left( \frac{1}{1 + (p/d)} \right) \,, \quad \mu = \mu_{f(H)\psi_0} \,. \tag{1.7}$$

The nonnegative reals $D_\mu^\pm(q)$ are called the generalized fractal dimensions of the measure $\mu$, and they are defined for $q \neq 1$ as follows:

$$D_\mu^-(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log I_\mu(q, \varepsilon)}{(q-1)\log\varepsilon} \quad \text{and} \quad D_\mu^+(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log I_\mu(q, \varepsilon)}{(q-1)\log\varepsilon} \,. \tag{1.8}$$

For general properties of these dimensions we refer to [BGT3, GT]. For the place of these dimensions in dynamical systems and in thermodynamics formalism, see [P].

Later on, (1.7) has been extended to any measure $\mu$ by one of us in [Tc1]. The lower bound (1.7) improves on previous ones (given in terms of Hausdorff or Packing dimension of $\mu$ [G, C, La, BCM, GSB1]), for in addition to be (i) non smaller, it allows (ii) for a non linear behavior in $p$: $\beta^\pm(p, f)$ may grow with $p$, where previous lower bounds were constant in $p$, and in addition it may be (iii) non zero for atomic measures (when previous ones were automatically zero in presence of pure point spectrum). In the present paper we present a one-dimensional general lower bound on the transport integrals $I_\mu(q, \varepsilon)$. Thanks to (1.5), it enables us to provide the first concrete applications of (1.7).

We shall prove the following (we refer to Theorem 2.1 for a precise statement): if $f \in \mathcal{C}_{c,+}^\infty(\mathbb{R})$, $f \geq 0$ and $f = 1$ on some set $S$ of positive Lebesgue measure, then for any $q \in (0, 1)$ and $\sigma > 0$,

$$I_{\mu_{f(H)\psi_0}}(q, T^{-1}) \geq C_q \, T^{1-q} \int_S \frac{k(E) \, \mathrm{d}E}{\|T(E, N, 0)\|^{2q}} - \frac{C_2}{T} \,, \quad T > 1 \,, \tag{1.9}$$

where $N = [T^{1+\sigma}]$ in the discrete case and $N = T^{1+\sigma}$ in the continuous case. Here $k(E)$ is some fixed finite function, positive for Lebesgue a.e. $E$ (in the discrete case $k(E) \equiv 1$). Combining (1.9) with (1.5) then provides a lower bound on the dynamics in terms of the behavior of transfer matrices. As a particular corollary, our result yields nontrivial lower bounds as soon as the transfer matrices $\|T(E,N,0)\|$ do not grow faster than polynomially in $N$ on a set of energies $E$ of positive Lebesgue measure, and this, regardless to the nature of the spectrum and regardless to the Hausdorff dimension of the spectral measure $\mu_{\psi_0}$. Indeed if one has $\|T(E,N,0)\| \leq C(E)N^\gamma$ for all $E \in S$, $|S| > 0$, and all $N$ large enough, then, if $f \equiv 1$ on $S$,

$$\beta^-(p,f) \geq 1 - \frac{2\gamma}{p}, \quad \beta^-(\infty,f) = 1, \tag{1.10}$$

with $\beta^-(\infty,f) = \lim_{p\to\infty} \beta^-(p,f)$. A similar bound follows for the upper transport exponents $\beta^+(p,f)$ if for some sequence of scales $N_i$, one checks $\|T(E,N_i,0)\| \leq C(E)N_i^\gamma$ for all $E \in S$. We refer to Theorem 2.2 and Theorem 2.3 for detailed statements.

We shall apply (1.10) and its analog for upper exponents to several kinds of potential $V$: a random decaying potential as considered in [KLS], the almost Mathieu operator as in [La, DR+2], and sparse potentials. In the last case we obtain dynamical bounds (with $\beta^-(\infty,f) = 1$ or $\beta^+(\infty,f) = 1$) for some bounded or unbounded potentials. Our application to the almost Mathieu operator provides, in particular, a new proof to the celebrated example of Del Rio, Jitomirskaya, Last, Simon [DR+2], where $\beta^+(p,f\equiv 1) = 1$ is shown to coexist with pure point spectrum and exponentially localized eigenstates. Indeed our analysis implies for this model that $D^+_{\mu_{\delta_1}}(q) = 1$, $q \in (0,1)$, and thus $\beta^+(\infty, f\equiv 1) = 1$ by (1.7). Thus, the general bound (1.7), which follows from Guarneri's old strategy, is powerful enough to take into account the mecanism which yields the quasi-ballistic dynamical behaviour pointed out in [DR+2]. It thereby sheds some new light on this famous example.

More generally, one of our goals in this paper is to provide a better understanding of the mechanism that can produce a non trivial transport even in presence of pure point spectrum. If traditionally, point spectrum has been associated with localized dynamics, the first example of a Schrödinger operator with point spectrum and unbounded growth of moments on a subsequence of times $t_n \to \infty$ of an initially localized state is this almost Mathieu operator mentioned above, coming from [DR+2]. Recently, an example of Schrödinger operator with point spectrum and $\beta^-(p,f\equiv 1) \geq 1-(2p)^{-1}$ has been studied in [JSBS]. By applying our criterion to the random decaying model of [KLS], we provide a new example of a concrete model with point spectrum, obtaining $\beta^-(\infty,f) = 1$ everywhere in the spectrum. The mechanism of transport in this example is different from both [DR+2] and [JSBS]. In [DR+2], the transport is fast only on a subsequence of times due to, roughly speaking, long periodic structures in the potential. In [JSBS], the fast transport is due to exceptional energies. If the support of $f$ does not contain such energies, the corresponding transport exponents vanish. In the example we discuss here, the lower bounds on the moments hold for all times and are valid everywhere in the spectrum.

As another corollary of our main result, we get the following. Let us assume that for some set $S$ of positive Lebesgue measure, one has

$$\sum_0^\infty x^\alpha \text{ess-inf}_S \|T(E,x,0)\|^{-2} = \infty \ , \tag{1.11}$$

where the symbol $\sum_0^\infty$ stands for the sum $\sum_{x=0}^\infty$ in the discrete case and the integral $\int_0^\infty \mathrm{d}x$ in the continuous case. Then as soon as $f \equiv 1$ on $S$, one has $\beta^+(p,f) \geq 1 - \frac{1+\alpha}{p}$ for all $p > 0$, and $\beta^+(\infty, f) = 1$. In particular, the case $\alpha = 0$ should be compared to Simon-Stolz criterion for absence of pure point spectrum [SiSt]. They show that if $\sum_0^\infty \|T(E,x,0)\|^{-2} = \infty$ then $E$ is not an eigenvalue. In other terms, if $\inf_{E\in S} \sum_0^\infty \|T(E,x,0)\|^{-2} = \infty$, then there is no point spectrum in $S$. But nothing can be said about transport. Here we require the same kind of condition but with the infimum inside the summation; then one can deduce not only the absence of point spectrum, but also non trivial transport: $\beta^+(p,E) \geq 1 - \frac{1}{p}$ on $S$.

If traditionally the most relevant order of the moment is the moment of order $p = 2$: $\mathbb{M}(2,f,T)$ (and its associated transport exponents $\beta^\pm(2,f)$), the other values of $p$ turn out to be meaningful, providing important information on the wave-packet structure. For instance, it is quite clear that wave-packets behave differently in cases where $\beta^\pm(p,f)$ is a constant $\beta^\pm(f)$, or if it does increase with $p$. The idea is that, in the first case, wave-packets do not spread out when travelling, and in the second case, different parts of a wave-packet travel at different speeds, so that wave-packets spread out when travelling. Natural quantities to look at are then the limits as $p$ goes to 0 or $\infty$: $\beta^\pm(0^+, f) = \lim_{p\to 0} \beta^\pm(p,f)$ and $\beta^\pm(\infty, f) = \lim_{p\to\infty} \beta^\pm(p,f)$. In this paper we shall relate the behavior of the moments $\mathbb{M}(p,f,T)$ to the speed of the different parts of the wave-packet. In particular we shall give a precise statement to the following idea: $\beta^\pm(0^+, f)$ gives information on the speed of the essential part of the wave-packets, while $\beta^\pm(\infty, f)$ gives information on the speed of the fastest part of the wave-packets. More precisely, set, for $\alpha \geq 0$,

$$P(\alpha, T) = \frac{2}{T} \int_0^\infty \mathrm{e}^{-2t/T} \left\| P_{(T^\alpha - 2)} \, \mathrm{e}^{-iHt} f(H) \psi_0 \right\|^2 \mathrm{d}t, \tag{1.12}$$

where $P_N$ is the spatial projection outside the ball of radius $N$ and center of the origin. Define $S^\pm(\alpha)$ the growth exponents of $P(\alpha, T)$. Then it is shown that

$$\beta^\pm(0^+, f) = \alpha_l^\pm \equiv \sup\{\alpha,\ S^\pm(\alpha) = 0\} \tag{1.13}$$

and

$$\beta^\pm(\infty, f) = \alpha_u^\pm \equiv \inf\{\alpha,\ S^\pm(\alpha) = +\infty\} \tag{1.14}$$

We moreover relate the behavior of the transfer matrices to the functions $S^\pm(\alpha)$ by showing: if $\|T(E,N,0)\| \leq N^\gamma$ for all $E \in S$, $|S| > 0$, and all $N$ large enough, then, if $f \equiv 1$ on $S$, one has $\alpha_u^- = 1$ and $S^-(\alpha) \leq 2\gamma$ for all $\alpha < \alpha_u^-$ (and a similar result for subsequences). The latter is a consequence of the lower bound described in (1.10). We refer to Section 4 for precise statements.

The paper is organized as follows. In Section 2 we state the general lower bounds we obtain for quantum dynamics, that we prove in Section 3 for both discrete and continuous models. In Section 4 we develop a general analysis of wave-packets that leads, in particular, to the characterization of the transport exponents as $p$ tends to 0 and $+\infty$. Section 5 is devoted to the application of our general lower bounds to Schrödinger operators with different type of potential: a random decaying potential (discrete and continuous model), several kind of sparse potential, the quasi-periodic potential studied in [La][DR+2] and that we revisit here. In Appendix we first prove the trace estimate one needs to apply [BGT1, BGT2] to the general class of potential we consider. We then provide a general approximation lemma that enters the proof of our main result in a crucial way, and that relies on the Helffer-Söjstrand formula.

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## 2 General lower bounds in dimension one

For $x \in \mathbb{Z}$ or $\mathbb{R}$ we shall use the following notations

$$\langle x \rangle = \sqrt{1+|x|^2} \quad \text{and} \quad (\langle X \rangle \psi)(x) = \langle x \rangle \psi(x),\ \psi \in \mathcal{H}. \tag{2.15}$$

The potential $V$ is assumed to be polynomially bounded: there exists $a, b > 0$ such that

$$|V(x)| \le a \langle x \rangle^b, \tag{2.16}$$

for all $x \in \mathbb{Z}^+$ in the discrete case, and $x \in \mathbb{R}^+$ in the continuous case. Moreover, in the continuous case we further suppose that the potential satisfies the following regularity property:

$$V = V^{(1)} + V^{(2)}, \quad \text{with} \quad 0 \le V^{(1)} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+, \mathrm{d}x), \tag{2.17}$$

and $V^{(2)}$ is relatively $-\Delta$ form-bounded with relative bound $< 1$. To fix the notation we thus require the existence of two constants $\Theta_1 < 1$ and $\Theta_2$ such that

$$\left| \langle \psi, V^{(2)} \psi \rangle \right| \le \Theta_1 \|\nabla \psi\| + \Theta_2 \|\psi\|^2, \text{ for all } \psi \in \mathcal{H}. \tag{2.18}$$

We note that our results extend to operators defined on the full line.

For a given operator $H$ on $\ell^2([1, +\infty))$, resp. $\mathrm{L}^2([0, +\infty))$, we define the transfer matrices $T(E, x, y)$ between sites $y$ and $x$ as:

$$T(E,x,y) = \begin{pmatrix} u_0(E,x+1) & u_{\pi/2}(E,x+1) \\ u_0(E,x) & u_{\pi/2}(E,x) \end{pmatrix}, \text{ resp. } \begin{pmatrix} u_0'(E,x) & u_{\pi/2}'(E,x) \\ u_0(E,x) & u_{\pi/2}(E,x) \end{pmatrix}, \tag{2.19}$$

where $u_\theta(E, x)$ denotes the solution of $Hu = Eu$, $E \in \mathbb{R}$, satisfying $u_\theta(E, y) = \sin\theta$, $u_\theta(E, y+1) = \cos\theta$, resp. $u_\theta(E, y) = \sin\theta$, $u_\theta'(E, y) = \cos\theta$ (note that $T(E, x, x) = \mathrm{Id}$). It follows from the definitions that if $u$ is a solution of the eigenvalue equation $Hu = Eu$, $E \in \mathbb{R}$, then

$$\begin{pmatrix} u(E,x+1) \\ u(E,x) \end{pmatrix} = T(E,x,y) \begin{pmatrix} u(E,y+1) \\ u(E,y) \end{pmatrix}, \text{ resp. } \begin{pmatrix} u'(E,x) \\ u(E,x) \end{pmatrix} = T(E,x,y) \begin{pmatrix} u'(E,y) \\ u(E,y) \end{pmatrix}. \tag{2.20}$$

Note that in the discrete case, the transfer matrix $T(E, x, y)$, $x > y \ge 0$, can be written as

$$T(E,x,y) = A(E,x) \cdots A(E,y+1), \qquad A(E,k) = \begin{pmatrix} E - V(k) & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.21}$$

We further define, for $E \in \mathbb{R}$, the measurable function

$$\gamma(E) = \limsup_{x\to+\infty} \frac{1}{\log x} \log \|T(E,x,0)\| \,,\ \gamma(E) \in [0,+\infty]. \tag{2.22}$$

Recall the definition of the moment of order $p$ given by (1.1) and its associated transport exponents given by (1.3). To specify transport rates nearby a given energy level, and following [GK2], we construct transport exponents associated to a given *open* interval $\beta^{\pm}(p,I)$ together with the local transport exponents $\beta^{\pm}(p,E)$ as

$$\beta^{\pm}(p,I) = \sup_{f\in\mathcal{C}_0^\infty(I)} \beta^{\pm}(p,f), \quad \beta^{\pm}(p,E) = \inf_{I\ni E} \beta^{\pm}(p,I). \tag{2.23}$$

We finally define the lower and upper asymptotic transport exponents at energy $E$ by

$$\beta^{\pm}(0^+,E) = \lim_{p\to 0} \beta^{\pm}(p,E) = \inf_p \beta^{\pm}(p,E) \in [0,1]\ . \tag{2.24}$$

$$\beta^{\pm}(\infty,E) = \lim_{p\to \infty} \beta^{\pm}(p,E) = \sup_p \beta^{\pm}(p,E) \in [0,1]\ . \tag{2.25}$$

Basic properties of such moments and transport exponents are studied in [GK2]. In particular the fact that the moments $\mathbb{M}(p,f,T)$ are finite and that the transport exponents defined in (2.23)-(2.25) lie in $[0,1]$ relies on [GK1]. It is valid for *any* potential $V$ in the lattice case, and under Conditions (2.17)-(2.18) in the continuum.

The key theoretical result we prove is the following.

**Theorem 2.1.** *Let $H = -\Delta + V$ where $V$ satisfies (2.16), and in addition (2.17)-(2.18) in the continuous case. Let $S$ be a bounded set of positive Lebesgue measure: $S \subset [-K,K]$. Pick $f \in \mathcal{C}^\infty_{c,+}(\mathbb{R})$, $f \geq 0$ and $f = 1$ on $S$. For any $q \in (0,1)$ and $\sigma > 0$ there exists constants $C_q > 0$ and $C_2$ (depending only on $q, f, \sigma, a, b, K$) such that for all $\varepsilon \in (0,1)$,*

$$I_{\mu_{f(H)\psi_0}}(q,\varepsilon) \geq C_q\, \varepsilon^{q-1} \int_S \frac{k^q(E)\,\mathrm{d}E}{\|T(E,N,0)\|^{2q}} - C_2\varepsilon\ , \tag{2.26}$$

*where $N = [\varepsilon^{-(1+\sigma)}]$ in the discrete case and $N = \varepsilon^{-(1+\sigma)}$ in the continuous case ; $k(E)$ is a finite constant, positive for Lebesgue a.e. $E$, given by*

$$\begin{cases} k(E) = 1 & \text{on } \mathcal{H} = \ell^2([1,+\infty)) \\ k(E) = K_{\Theta_1,\Theta_2} \frac{|\langle u_0(E),\chi_0\rangle|^2}{1+|E|} & \text{on } \mathcal{H} = \mathrm{L}^2([0,+\infty)), \end{cases} \tag{2.27}$$

*where the constant $K_{\Theta_1,\Theta_2} > 0$ depends only on $\Theta_1, \Theta_2$ appearing in (2.18), and $u_0$ is defined below Eq. (2.19). As a consequence, for any $p > 0$ and $T > 0$,*

$$\mathbb{M}(p,f,T) \geq C_p\, T^p \left( \frac{1}{\log T} \int_S \frac{k(E)^{\frac{1}{p+1}}\,\mathrm{d}E}{\|T(E,N,0)\|^{\frac{2}{p+1}}} \right)^{p+1} - C_3, \tag{2.28}$$

*with $N = [T^{1+\sigma}]$ in the discrete case and $N = T^{1+\sigma}$ in the continuous case. The constant $C_3$ depends on $p, f, \sigma, a, b, K$. The constants $C_p > 0$ and $C_3$ in (2.28) depend also on $\Theta_1, \Theta_2$ in the continuous case.*

**Remark 2.1.** *(i) Theorem 2.1 is stated for a given set $S$ at a given time $T$ (and thus a given scale $N$). This flexibility will thus allow for different kind of applications. For instance $S$ may depend on $T$, or one might consider time sequences to get result on upper exponents.*
*(ii) We note that the conclusions of Theorem 2.1 do not depend on the nature of the spectral measure $\mu_{f(H)\psi_0}$. That is why our result enables one to investigate indifferently dynamics in the pure point or singular continuous region. Moreover, it is easy to see that the bound (2.28) remains stable under finite perturbations of operator $H = -\Delta + V$. Indeed, one can easily see that if one changes the potential $V$ in a compact region then, in the same sense as in (1.6), $\|T'(E,N,0)\| \sim \|T(E,N,0)\|$, where $T$ and $T'$ denote, respectively, the transfer matrices for the unperturbed and the perturbed Hamiltonian. The constants in the equivalence are uniform in $N > 0$ and $E \in I$, $I$ a compact interval. As a consequence, the two corresponding transport integrals in (2.28) are equivalent. At the same time it is well known that changing the potential even in one point may change dramatically the nature of the spectrum (leading, for example, to a transition from pure point to singular continuous, e.g. [DRMS]) and the support of the spectral measure. Thus, from this point of view, dynamical results are more stable than spectral ones. This stability holds, of course, for all the results of the present paper since they are obtained using Theorem 2.1. Such an observation was already made in a similar context in [DT].*
*(iii) The result immediately extends to any boundary condition at the origin (provided the operator is self-adjoint). One then has to change the solution $u_0$ accordingly in the constant $k(E)$ in (2.27).*
*(iv)To keep the size of this paper in check, we decided to discuss in detail half-line case only. There is a straightforward extension to the whole line case, where $\|T(E,N,0)\|$ in (2.26) and (2.28) is replaced by $\min(\|T(E,N,0)\|, \|T(E,-N,0)\|)$. The proofs are nearly identical; a small adaptation is needed in the proof of Lemma 3.1 which we will leave to the interested reader. We are going to use the whole line version when discussing transport in the almost Mathieu equation (Section 5.3).*

Theorem 2.1 is a combination of the three following ingredients :
(i) The general bound obtained in [BGT1, BGT2], which provides a lower bound for the moments of order $p$ and at time $T$ using transport integrals $I_\mu(q,\varepsilon)$, with $q = (1+\frac{p}{d})^{-1}$ and $\varepsilon = T^{-1}$ (here $d = 1$).
(ii) The equivalence property between the transport integral $I_\mu(q,\varepsilon)$ and the generalized Rényi integral $L_\mu(q,\varepsilon) = \frac{1}{\varepsilon}\int \mathrm{d}x \mu(x-\varepsilon, x+\varepsilon)^q$ proved in [BGT3].
(iii) The lower bound on the spectral measure, evaluated on a ball of radius $\varepsilon = T^{-1}$, in terms of the behavior of the transfer matrices $T(E,N,0)$ with $N \approx T$, and for almost all energy with respect to the *Lebesgue* measure (and not to the spectral measure!). Such a lower bound is given by Proposition 2.1 below.

Proposition 2.1 indeed converts the upper bound on $\|T(E,n,0)\|$ into a lower bound on the spectral measure. The idea of the proof is similar to that in [CM, Section 4] where $\dim_P(\mu) = 1$ is proved for sparse barriers model. Our proof is however technically different, and provides more precise estimates in terms of the $\|T(E,n,0)\|$'s.

**Proposition 2.1.** *Let $H$ be as in Theorem 2.1, $\psi_0$ as in (1.2), and let $I$ be a compact interval. There exist a universal constant $C_1$ and for all $M > 0$ and $\sigma > 0$, a constant $C_2$ (depending on $I, M, \sigma, a, b$) such that for all $\varepsilon \in ]0,1[$ and all $\lambda \in I$, one has (setting*

*$N = [\varepsilon^{-1-\sigma}]$ in the discrete case and $N = \varepsilon^{-1-\sigma}$ in the continuum)*

$$\mu_{\psi_0}(\lambda - \varepsilon, \lambda + \varepsilon) \geq C_1 \int_{\lambda - \frac{\varepsilon}{2}}^{\lambda + \frac{\varepsilon}{2}} \frac{k(E)\,\mathrm{d}E}{S(N,E)} - C_2 \varepsilon^M \geq C_1 \int_{\lambda - \frac{\varepsilon}{2}}^{\lambda + \frac{\varepsilon}{2}} \frac{k(E)\,\mathrm{d}E}{\|T(E,N,0)\|^2} - C_2 \varepsilon^M. \tag{2.29}$$

*Here $S(N,E) = u_0^2(N,E) + u_0^2(N+1,E)$ in the discrete case, $S(N,E) = (u_0(N,E))^2 + (u_0'(N+1,E))^2$ in the continuous case, and $k(E)$ is given in Theorem 2.1 Eq. (2.27).*

**Remark 2.2.** *(i) While in the present paper we use only the lower bound (2.29), the upper bound*

$$\mu_{\delta_1}(\lambda - \varepsilon, \lambda + \varepsilon) \leq C_1^{-1} \int_{\lambda - 2\varepsilon}^{\lambda + 2\varepsilon} \|T(E,N,0)\|^2 \,\mathrm{d}E + C_2 \varepsilon^M, \tag{2.30}$$

*can easily be derived from our analysis in the discrete case.*
*(ii) It is clear that in the case of singular measures the first bound in (2.29) may be significantly better for some energies $\lambda$. Indeed, if the norm $||T(E,N,0)||$ is large for some $E$, it is possible that $S(N,E)$ is small (of order $||T(E,N,0)||^{-2}$). This may happen if $E$ is close to the spectrum of $H$. However, to use the stronger bound, one should have a rather good control from above of $S(N,E)$ for a given large $N$ as a function of $E$, which is not easy.*
*(iii) The power 2 of $\|T(E,N,0)\|$ in Proposition 2.1 is optimal, as can be seen, for instance, from the analysis on discrete sparse potential achieved in [Tc2].*
*(iv) As an illustration, assume that for Lebesgue a.e. $E$ in a neighborhood of $\lambda$, one has $\|T(E,N,0)\| \leq CN^\gamma$, where $C$ is uniform in $E$ and $N$. The bound (2.29) yields $\mu(\lambda - \varepsilon, \lambda + \varepsilon) \geq C' \varepsilon^{1+2\gamma+2\sigma}$. If as far as spectral dimensions are concerned such a lower bound is useless, it turns out to be quite useful for transport properties, as already noticed in [Tc1]. In addition, note that (2.30) yields, in this particular case, $\mu(x-\varepsilon, x+\varepsilon) \leq C_\nu \varepsilon^{1-2\gamma-2\sigma}$, for any $\sigma > 0$, and thus $\mu$ is $1-2\gamma$ continuous if $\gamma < 1/2$.*

We turn to some consequences of Theorem 2.1. Let $S$ be any Borel set with $\mu(S) > 0$, and $f$ a measurable function. Define $f_S = \mu\text{-essinf}_S f(E)$. Let us recall that $f_S$ is the unique real number such that one has simultaneously:
(i) $f(E) \geq f_S$ for $\mu$-a.e. $E$,
(ii) for all $\nu > 0$, there exists $S_\nu \subset S$, $\mu(S_\nu) > 0$, such that for all $E \in S_\nu$, one has $f(E) \leq f_S + \nu$.

**Theorem 2.2.** *Let $H$ be as in Theorem 2.1, $\psi_0 \in \mathcal{H}$ as in (1.2), and recall (2.22). Suppose there exists a bounded Borel set $S \subset [-K,K]$, of positive Lebesgue measure: $|S| > 0$, such that $\gamma_S = \text{Leb-essinf}_S \gamma(E) < \infty$. Then, for all $f \in \mathcal{C}^\infty_{c,+}(\mathbb{R})$, $f = 1$ on $S$, one has for all $p > 0$,*

$$D^-_{\mu_{f(H)\psi_0}}((1+p)^{-1}) \geq 1 - \frac{2\gamma_S}{p} \ . \tag{2.31}$$

*For the moment $\mathbb{M}(p,f,T)$ itself, for any $\nu > 0$*

$$\mathbb{M}(p,f,T) \geq C_1 T^{p - 2\gamma_S - \nu} - C_2 \tag{2.32}$$

*where $C_1 > 0$ depends on $p, \nu$ and $C_2 > 0$ depends on $p, f, \nu, a, b, K$. These constants depend also on $\Theta_1$, $\Theta_2$ in the continuous case.*

*It follows that if $\overline{\gamma}(E) = \sup_{\delta>0} \gamma_{]E-\delta,E+\delta[} < \infty$, then for all $p > 0$,*

$$\beta^-(p, E) \geq 1 - \frac{2\overline{\gamma}(E)}{p} \text{ , and thus } \beta^-(\infty, E) = 1. \tag{2.33}$$

*(Note that if $\gamma(E)$ is continuous at $E$, then $\overline{\gamma}(E) = \gamma(E)$.)*

Let us comment this result. First of all, of course, (2.32) starts to be nontrivial for $p > 2\gamma_S$. Next, in the case of the whole line operator, it is sufficient to assume that one takes in (2.22) either the limit $x \to +\infty$ or $x \to -\infty$ for all conclusions to remain true.

In the particular case where $\gamma(E) = 0$ for Lebesgue a.e. $E \in I$, where $I$ is some open interval, Theorem 2.2 asserts that $\beta^-(p, E) = 1$ on $I$. One may see this as the dynamical version of spectral results saying that if the transfer matrices are bounded, then the spectrum on $I$ has an absolutely continuous component [Si1], which implies $\beta^-(p, E) = 1$ by Guarneri's arguments [G]. Actually the absolutely continuous spectrum gives a little bit more: it gives $\mathbb{M}(p, f, T) \geq CT^p$ in (2.32). But on the other hand (2.22) with $\gamma(E) = 0$ is a weaker condition than the strict boundedness of the transfer matrices.

More generally, Theorem 2.2 belongs to the set of results relating power law upper bounds for the transfer matrix and lower bounds for dynamics. The oldest results are obtained using the power-law subordinacy theory of Jitomirskaya-Last. Provided $\gamma(E) \leq \gamma < 1/2$ on some set $S$, it ensures that the spectral measure $\mu_{\delta_1}$ restricted to $S$ is, if not zero, $1-2\gamma$ continuous ([JL, Corollary 4.4]); it then yields through Guarneri's type argument as developed by Combes [C] and Last [La], $\mathbb{M}(p, f, T) \geq C(J, \nu)T^{p(1-2\gamma)}$. This bound is nontrivial for all $p > 0$, provided $\gamma < 1/2$ (the condition $\gamma < 1/2$ prevents the spectrum from having eigenvalues). If the set $S$ has a positive Lebesgue measure, one can compare it with the bound of Theorem 2.2. One can easily see that the bound (2.32) is better for $p > 1$ whatever is the value of $\gamma \in (0, 1/2)$. Strictly speaking, $\mu_{\delta_1}$, as a singular measure, may be supported outside $S$, in which case the subordinacy theory cannot supply any information at all. On the other hand, if $|S| = 0, \mu_{\delta_1}(S) > 0$, then our Theorem 2.2 cannot be applied directly.

Recently, Damanik and Tcheremchantsev [DT] have obtained dynamical lower bounds in the case where

$$\|T(E, n, 0)\| \leq CN^\gamma, \ |n| \leq N \tag{2.34}$$

with constants $C, \gamma$ uniform in $N, E$ on some set $A(N)$ of energies depending on $N$. The proof (which holds only in the discrete case and for the initial state $\psi = \delta_1$) is based on Parseval formula and is completely different from that of Theorems 2.1, 2.2. Surprisingly, it is sufficient that $A(N)$ consists of a single energy $E_c$ independent of $N$, to have nontrivial dynamical lower bound. The method in [DT] is good if the sets $A(N)$ are "thin", but is far from being optimal in general. In particular, under the conditions of Theorem 2.2, one can show that the methods of [DT] give the bound for the moments like $T^{(p-3\gamma_S)/(1+\gamma_S)-\nu}$ for any $\nu > 0$, which is weaker than (2.32).

When comparing Theorems 2.1 and 2.2, one can observe that Theorem 2.1 has a wider domain of applications than Theorem 2.2. For example, if there is only a single energy (or a finite number of energies) $E_c$ where $\gamma(E) < +\infty$, Theorem 2.2 gives no result. At the same time in such cases Theorem 2.1 may work. In particular, this is the case of random polymer model [JSBS], where one proves that $||T(E, N, 0)|| \leq C$ for

$E : |E-E_c| \leq C/\sqrt{N}$. It is easy to see that Theorem 2.1 yields $\beta^-(p,1) \geq \frac{1}{2}(1-1/p)$ if $f=1$ near $E_c$. This bound is, however, weaker than that of [JSBS] $\beta^-(p,1) \geq 1-1/(2p)$ obtained by methods of [DT].

We have the following equivalent of Theorem 2.2 for subsequences of time.

**Theorem 2.3.** *Let $H$ be as in Theorem 2.1 and $\psi_0 \in \mathcal{H}$ as in (1.2). For a given increasing sequence $(n_i)_{i\geq 0}$ such that $\lim_{i\geq 0} n_i = +\infty$, we define for $E \in \mathbb{R}$ the measurable function taking values on $[0,+\infty]$:*

$$\gamma(E) = \limsup_{i\to\infty} \frac{\log||T(E,n_i,0)||}{\log n_i}. \tag{2.35}$$

*Suppose there exists a bounded Borel set $S \subset [-K,K]$, of positive Lebesgue measure: $|S|>0$, such that $\gamma_S = \text{Leb-essinf}_S \gamma^{(n_i)}(E) < \infty$. Then, for all $f \in \mathcal{C}^\infty_{c,+}(\mathbb{R})$, $f=1$ on $S$, one has for all $p>0$,*

$$D^+_{\mu_{f(H)\psi_0}}((1+p)^{-1}) \geq 1 - \frac{2\gamma_S}{p}\ . \tag{2.36}$$

*For the moment $\mathbb{M}(p,f,T)$ itself, for any $\nu>0$ and $i \geq 0$,*

$$\mathbb{M}(p,f,T_i) \geq C_1 T_i^{p-2\gamma_S-\nu} - C_2. \tag{2.37}$$

*It follows that if $\overline{\gamma}(E) = \sup_{\delta>0}\gamma_{[E-\delta,E+\delta]} < \infty$, then for all $p>0$,*

$$\beta^+(p,E) \geq 1 - \frac{2\overline{\gamma}(E)}{p}\ ,\ \textit{and thus } \beta^+(\infty,E)=1. \tag{2.38}$$

**Corollary 2.1.** *Let $H$ be as in Theorem 2.1. Suppose that for some set $S$ of positive Lebesgue measure, and for some $\alpha \geq 0$,*

$$\sum_0^\infty x^\alpha \,\text{ess-inf}_S \left(\frac{1}{\|T(E,x,0)\|^2}\right) = \infty\ , \tag{2.39}$$

*where the symbol $\sum_0^\infty$ stands for the sum $\sum_{x=0}^\infty$ in the discrete case and the integral $\int_0^\infty \mathrm{d}x$ in the continuous case, then for any $E \in S$, $\beta^+(p,E) \geq 1 - \frac{1+\alpha}{p}$ for all $p>0$, and $\beta^+(E)=1$.*

# 3 Proof of the general bounds

## 3.1 Proof of the spectral bounds: the lattice case

This subsection is devoted to the proof of Proposition 2.1. Let $E_0 \in \mathbb{R}$. We introduce "finite volume operators" by cutting the potential after some site $N>0$ and replace it by a constant:

$$H^{(E_0,N)} = -\Delta + V\chi_{[0,N]} + E_0(1-\chi_{[0,N]}).$$

The constant $E_0$ will be chosen so that approximating operator has bounded solutions at the energy interval of interest to us. In particular, if this interval does not lie in the spectrum of free operator, the shift will be necessary. We write $\mu_\psi^{(E_0,N)}$ for the spectral measure associated to $\psi$ and $H^{(E_0,N)}$, and $R^{(E_0,N)}(z) = (H^{(E_0,N)}-z)^{-1}$ for the corresponding resolvent. We prove the following technical result.

**Lemma 3.1.** *Let $H = -\Delta + V$ be any discrete Schrödinger operator. There exists a finite universal constant $C_1 > 0$, such that for any $E_0 \in \mathbb{R}$, $E \in [E_0 - 1, E_0 + 1]$ and $N > 1$,*

$$\frac{\mathrm{d}\mu_{\delta_1}^{(E_0,N)}}{\mathrm{d}x}(E) \geq \frac{C_1}{u_0^2(N,E) + u_0^2(N+1,E)} \geq \frac{C_1}{\|T(E,N,0)\|^2}. \tag{3.1}$$

*Here $T(E,N,0)$ is the transfer matrix of $H^{(E_0,N)}$, which coincides with the same transfer matrix for the original operator $H$.*

**Remark 3.1.** *In its spirit, Lemma 3.1 is close to a result of Simon in [Si1]. The bound in [Si1] is derived for general 1D discrete and continuous Schrödinger operators provided the transfer matrices are uniformly bounded in $N$. However, the power that one gets in this general context is not as good as the one Lemma 3.1 provides for the particular operator $H^{(E_0,N)}$: one gets $\sup_N \|T(E,N,0)\|^6$ in [Si1] and $\|T(E,N,0)\|^2$ here. It is of course crucial for the dynamical lower bound to get the smallest possible power. A related result in a continuous setting has been used already by Pearson [P1] in his work on sparse potentials.*

**Proof of Lemma 3.1:** Let $E_0$ and $E$ be as in the lemma. It follows from the Stone's formula that

$$\begin{aligned}\frac{\mathrm{d}\mu_{\delta_1}^{(E_0,N)}}{\mathrm{d}x}(E) &= \frac{1}{\pi}\lim_{\eta\to 0} \mathrm{Im}\langle \delta_1,\, R^{(E_0,N)}(E+i\eta)\delta_1\rangle && (3.2)\\ &= \frac{1}{\pi}\lim_{\eta\to 0} \eta \|R^{(E_0,N)}(E+i\eta)\delta_1\|^2. && (3.3)\end{aligned}$$

Let $\varphi = R^{(E_0,N)}(E+i\eta)\delta_1,\ \eta > 0$. We estimate $\|R^{(E_0,N)}(E+i\eta)\delta_1\|^2$ from below as follows:

$$\|R^{(E_0,N)}(E+i\eta)\delta_1\|^2 \geq \frac{1}{2}\sum_{n=N}^{+\infty}\left(|\varphi(n)|^2 + |\varphi(n+1)|^2\right). \tag{3.4}$$

Using the transfer matrices representation, one has for any $n \geq N$,

$$\begin{aligned}|\varphi(N)|^2 + |\varphi(N+1)|^2 &= \left\|T^{-1}(E+i\eta,n,N)(\varphi(n+1),\varphi(n))^T\right\|^2\\ &\leq \|T(E+i\eta,n,N)\|^2\left(|\varphi(n)|^2 + |\varphi(n+1)|^2\right), && (3.5)\end{aligned}$$

where $T$ is the transfer matrix corresponding to the operator $H^{(E_0,N)}$. In (3.5) we used that $\|T^{-1}\| = \|T\|$ for any $2 \times 2$ matrix with complex coefficients and determinant 1 (see e.g. [CFKS], chapter 9). Note that $T(E+i\eta,n,N) = T_0(E-E_0+i\eta,n,N)$ for $n \geq N$, where $T_0$ is the transfer matrix of the free Laplacian:

$$T_0(z,n,m) = A_0(z)^{n-m}, \qquad A_0(z) = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that for any real $w \in [-1,1]$, $T_0(w,n,m) = (A_0(w))^{n-m}$ with $A_0(w)$ elliptic, so that

$$\sup_{|w|\leq 1}\sup_{n,m\in\mathbb{Z}} \|T_0(w,n,m)\| = C\,, \tag{3.6}$$

where $C>0$ is a finite universal constant. As one leaves the real line for the complex plane the bound (3.6) no longer holds. However, for any $|w|\leq 1,\ |\eta|\leq 1$ one can show that the following bound still holds:

$$\|T_0(w+i\eta,n,m)\|\leq C(1+C\eta)^{n-m}. \tag{3.7}$$

Indeed, $A_0(w+i\eta)=A_0(w)+i\eta J$, where $J=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, $\|J\|=1$, so that developing $T_0(w+i\eta,n,m)=A_0(w+i\eta)^{n-m}$ yields (3.7). As a consequence, plugging (3.5) and (3.7) into (3.4) gives

$$\begin{aligned}\eta\|R^{(E_0,N)}(E+i\eta)\delta_1\|^2 &\geq \frac{\eta}{2C^2}\left(|\varphi(N)|^2+|\varphi(N+1)|^2\right)\sum_{n=N}^{+\infty}(1+C\eta)^{-2(n-N)}\\ &\geq \frac{1}{4C^3}\left(|\varphi(N)|^2+|\varphi(N+1)|^2\right). \end{aligned}\tag{3.8}$$

It is well known that

$$\varphi(N)\equiv(R^{(E_0,N)}(E+i\eta)\delta_1)(N)=u_{\pi/2}(N,E+i\eta)+m(E+i\eta)u_0(N,E+i\eta),$$

where $m(z)$ is the Weyl function of operator $H^{(E_0,N)}$ (the Borel transform of its spectral measure). One observes that solutions $u_0(n,E),\ u_{\pi/2}(n,E)$ for $n\leq N+1$ are the same for both $H^{(E_0,N)}$ and $H$ since the potentials coincide on $[0,N]$. Since the measure of $H^{(E_0,N)}$ is absolutely continuous and $u_0(N,E),u_{\pi/2}(N,E)$ are both real, we have

$$\lim_{\eta\to 0}|\varphi(N)|^2\geq(u_{\pi/2}(N,E)+\operatorname{Re}m(E+i0)u_0(N,E))^2 \tag{3.9}$$

where $m(E+i0)$ is finite and similarly for $\varphi(N+1)$. It follows from (3.3), (3.8) and (3.9) that

$$\begin{aligned}\frac{\mathrm{d}\mu_{\delta_1}^{(E_0,N)}}{\mathrm{d}x}(E)\geq C((u_{\pi/2}(N,E)+\operatorname{Re}m(E+i0)u_0(N,E))^2+\\ (u_{\pi/2}(N+1,E)+\operatorname{Re}m(E+i0)u_0(N+1,E))^2).\end{aligned}\tag{3.10}$$

Straightforward computations show that the minimum of the polynomial $t\to(a+tb)^2+(c+td)^2$ is obtained for the particular value $t_0=-(ab+cd)/(b^2+d^2)$. Putting that into (3.10) and using the fact that the Wronskian of $u_0$ and $u_{\pi/2}$ is one, we get the first bound of (3.1). The second bound of (3.1) follows directly since $(u_0(N+1,E),u_0(N,E))^T=T(E,N,0)(1,0)^T$. □

**Proof of Proposition 2.1:** For given $\lambda\in I,\ N$ we shall use Lemma A.1 with $H_1=H$ and $H_2=H^{(\lambda,N)}$. Since $I$ is compact and the potential $V$ is polynomially bounded, one can see that $|V_1(x)-V_2(x)|\leq A\langle x\rangle^b$ with constants $A,b$ uniform in $\lambda\in I,N$. Thus, the bound of Lemma A.1 together with Lemma 3.1 gives for any $M>0,\sigma>0$,

$$\mu_{\delta_1}(\lambda-\varepsilon,\lambda+\varepsilon)\ \geq\ \mu_{\delta_1}^{(\lambda,N)}(\lambda-\frac{\varepsilon}{2},\lambda+\frac{\varepsilon}{2})-C_2\varepsilon^M \tag{3.11}$$

$$\geq\ \int_{\lambda-\frac{\varepsilon}{2}}^{\lambda+\frac{\varepsilon}{2}}\frac{C_1\,\mathrm{d}E}{S(N,E)}-C_2\varepsilon^M, \tag{3.12}$$

where $N=[\varepsilon^{-1-\sigma}]$ and $C_2<\infty$ depends only on $I,M,\sigma,a,b$. □

### 3.2 Proof of the spectral bounds: the continuous case

Following [GK1], the approximation lemma given in the appendix remains valid in the continuum as well (in particular one needs a suitable Combes-Thomas estimate with explicit rate of exponential decay). We are thus left with the proof of the analog of Lemma 3.1 in the continuum. For the sake of completeness we provide a sketch of the argument. We show that there exist constants $k(E) > 0$ for Lebesgue a.e. $E \in \mathbb{R}$, such that if $E_0$ is such that $E \in [E_0 + 1, E_0 + 3]$, then

$$\frac{\mathrm{d}\mu_{\chi_0}^{(E_0,N)}}{\mathrm{d}x}(E) \geq \frac{k(E)}{\|T(E,N,0)\|^2} \,. \tag{3.13}$$

It then follows from the latter that for $\lambda \in \mathbb{R}$, and for any $\varepsilon \in (0,1)$,

$$\mu_{\chi_0}^{(\lambda-2,N)}(\lambda-\varepsilon,\lambda+\varepsilon) \geq \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \frac{k(E)\mathrm{d}E}{\|T(E,N,0)\|^2} \,.$$

To prove (3.13) we proceed as in the discrete case, namely we cut the potential and use the matrices of the free Laplacian after the cut. We first make use of Sobolev-estimates. Recall $V = V^{(1)} + V^{(2)}$, where $V^{(1)} \geq 0$ and $V^{(2)}$ is relatively $-\Delta$ bounded with nonnegative constants $\Theta_1 < 1$ and $\Theta_2$ as in (2.18). As a consequence there exists a constant $K_{\Theta_1,\Theta_2} < \infty$, depending only on $\Theta_1, \Theta_2$, such that uniformly in $n \geq 1$,

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\varphi'(x)|^2 \mathrm{d}x \leq K_{\Theta_1,\Theta_2}(1+|E|) \int_{n-1}^{n+1} |\varphi(x)|^2 \mathrm{d}x \tag{3.14}$$

(see e.g. [CFKS] or [GK2, Lemma A.2]). For convenience we assume $K_{\Theta_1,\Theta_2} \geq 1$ (by may be enlarging $K_{\Theta_1,\Theta_2}$ if necessary). Pick $\eta \leq 1$, and consider the vector

$$\varphi = R^{(E_0,N)}(E+i\eta)\chi_0 \,. \tag{3.15}$$

It follows from (3.14) that

$$\begin{aligned}
\eta\|\varphi\|^2 &\geq \frac{\eta}{2}\int_{x>N} |\varphi(x)|^2\mathrm{d}x \;+\; \frac{\eta}{4}\sum_{k\geq N}\int_{k-1}^{k+1} |\varphi(x)|^2\mathrm{d}x \\
&\geq \frac{\eta}{4K_{\Theta_1,\Theta_2}(1+|E|)}\int_{x>N} (|\varphi(x)|^2 + |\varphi'(x)|^2)\mathrm{d}x \\
&\geq \frac{|\varphi(N)|^2+|\varphi'(N)|^2}{4K_{\Theta_1,\Theta_2}(1+|E|)}\int_{x>N} \frac{\eta\,\mathrm{d}x}{\|T(E-E_0+i\eta,x,N)\|^2} \\
&\geq \frac{\tilde{K}_{\Theta_1,\Theta_2}}{1+|E|}\left(|\varphi(N)|^2+|\varphi'(N)|^2\right),
\end{aligned} \tag{3.16}$$

where we used the boundedness of the transfer matrix $T(E,N,x) = T_0(E-E_0,N,x)$, with $(E-E_0) \in [1,3]$, $x > N$, and $T_0$ the free transfer matrix, and the continuous analog of (3.7) (e.g. [Si1]). The constant $\tilde{K}_{\Theta_1,\Theta_2} > 0$ is independent of $E$. Using now the kernel formula for the resolvent $R^{(E_0,N)}(z)$, one gets an expression for $\varphi$ in terms of the basic solutions $u_0$ and $u_{\pi/2}$ at complex energy $z = E + i\eta$:

$$\begin{aligned}
\varphi(N) &= u_0(N,z)\int_N^{\infty} f(y,z)\chi_0(y)\mathrm{d}y + f(N,z)\int_0^N u_0(y,z)\chi_0(y)\mathrm{d}y \\
&= f(N,z)\int_0^1 u_0(y,z)\mathrm{d}y,
\end{aligned}$$

where $f(x,z) = u_{\pi/2}(x,z) + m(z)u_0(x,z)$, and $m$ is the Weyl function (see e.g. [T]). Similarly, $\varphi'(N) = f'(N,z)\int_0^1 u_0(y,z)\mathrm{d}y$. Set $c_0(z) = \langle u_0, \chi_0\rangle = \int_0^1 u_0(y,z)\mathrm{d}y$. Letting $\eta$ going to zero, it thus follows from (3.16) (see the proof of the discrete case for more details) that

$$\frac{\mathrm{d}\mu_{\chi_0}^{(E_0,N)}}{\mathrm{d}x}(E) \geq \frac{\tilde{K}_{\Theta_1,\Theta_2}|c_0(E)|^2}{1+|E|}\left\{(u_{\pi/2}(N,E) + \mathrm{Re}\, m(E+i0)u_0(N,E))^2 \right. \\ \left. + (u'_{\pi/2}(N,E) + \mathrm{Re}\, m(E+i0)u'_0(N,E))^2\right\}. \tag{3.17}$$

The function $c_0(z)$ is analytic by analyticity of $\varphi(x,z)$ (see e.g. [T]), thus $c_0(E)$ is not zero for Lebesgue almost every $E$. The r.h.s. of (3.17) can be now bounded from below in the same spirit as in (3.10) in the discrete case. Finally, we get

$$\frac{\mathrm{d}\mu_{\chi_0}^{(E_0,N)}}{\mathrm{d}x}(E) \geq \frac{k(E)}{|u_0(N,E)|^2 + |u'_0(N,E)|^2} \geq \frac{k(E)}{\|T(E,N,0)\|^2}\,.$$

Here $k(E) = \tilde{K}_{\Theta_1,\Theta_2}(1+|E|)^{-1}|c_0(E)|^2$ is positive for Lebesgue almost $E$. $\square$

### 3.3 Proof of the lower bounds on the moments

We first prove Theorem 2.1. It will be a combination of Proposition 2.1 and the lower bound on the dynamics given by the transport integrals and that relies on [BGT1, BGT2, BGT3].

**Proof of Theorem 2.1:**
Let $f \in \mathcal{C}_0^\infty(\mathbb{R})$, $f \geq 0$, $f = 1$ on $S$. We first derive the lower bound (2.26) on transport integrals

$$I_\mu(q,\varepsilon) = \int_{\mathrm{supp}\mu} \mathrm{d}\mu(x)\ \mu(x-\varepsilon, x+\varepsilon)^{q-1} \quad \text{with } \mu = \mu_{f(H)\psi_0}\,. \tag{3.18}$$

We shall take advantage of the equivalence of different definitions of the generalized fractal dimensions of a measure, as stated in [BGT3, Theorem 2.1]. Such an equivalence was well known, and rather trivial, for $q > 1$. It is not the case in the more delicate regime $q \in (0,1)$, which has recently been treated in [BGT3] for any Borel measure on $\mathbb{R}$ of finite mass. In particular putting together Lemma 2.1 and Lemma 2.3 of [BGT3] gives, for some finite geometric constant $C_q > 0$, and for any Borel measure $\mu$ of finite mass,

$$\frac{C_q}{\varepsilon}\int_{\mathbb{R}} \mathrm{d}x\ \mu(x-\varepsilon,x+\varepsilon)^q \leq I_\mu(q,\varepsilon) \leq \frac{C'_q}{\varepsilon}\int_{\mathbb{R}} \mathrm{d}x\ \mu(x-\varepsilon,x+\varepsilon)^q. \tag{3.19}$$

We thus have to bound from below the quantity $\frac{1}{\varepsilon}\int_{\mathbb{R}} \mathrm{d}x\left(\mu_{f(H)\delta_1}(x-\varepsilon,x+\varepsilon)\right)^q$. Since the function $f$ is uniformly continuous, there exists $\eta \in (0,1)$ such that $|f(x)-f(y)| \leq \frac{1}{2}$ provided $|x-y| \leq 2\eta$. Define the set

$$J = \{x \mid \mathrm{d}(x,S) \leq \eta\}.$$

Now for any $\varepsilon < \eta$ and for any $x \in J$, one verifies that $(x-\varepsilon, x+\varepsilon) \subset \{y,\ \mathrm{d}(y,S) < 2\eta\} \subset \{y,\ f(y) \geq \frac{1}{2}\}$ (recall that $f = 1$ on $S$). It follows that for all $\varepsilon < \eta$ and $x \in J$

$$\mu_{f(H)\delta_1}(x-\varepsilon, x+\varepsilon) = \int_{x-\varepsilon}^{x+\varepsilon} f(y)\mathrm{d}\mu_{\delta_1}(y) \geq \frac{1}{2}\mu_{\delta_1}(x-\varepsilon, x+\varepsilon). \tag{3.20}$$

Moreover, for any given $M > 0$ and $\sigma > 0$, we get from Proposition 2.1, Eq. (2.29) that uniformly in $x \in I = [-B-1, B+1]$, where $B$ is such that $\mathrm{supp} f \subset [-B, B]$,

$$\mu_{\delta_1}([x-\varepsilon, x+\varepsilon]) \geq A(x,\varepsilon) - D(\varepsilon),$$

with

$$A(x,\varepsilon) = C_1 \int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} \frac{k(E)\mathrm{d}E}{\|T(E,N,0)\|^2}, \quad D(\varepsilon) = C_2\, \varepsilon^M.$$

Recall the elementary inequality $(A-D)^q \geq A^q - D^q$, where $q \in (0,1),\ A \geq D$. Since $\mu_{\delta_1}([x-\varepsilon, x+\varepsilon]) \geq 0$, we thus always have

$$\mu_{\delta_1}([x-\varepsilon, x+\varepsilon])^q \geq A(x,\varepsilon)^q - D(\varepsilon)^q\ . \tag{3.21}$$

Moreover it follows from Jensen inequality that:

$$A(x,\varepsilon)^q = \left( \int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} \frac{\varepsilon C_1 k(E)}{\|T(E,N,0)\|^2} \frac{\mathrm{d}E}{\varepsilon} \right)^q \geq C_1^q \varepsilon^{q-1} \int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} \frac{k(E)^q(E)\mathrm{d}E}{\|T(E,N,0)\|^{2q}}. \tag{3.22}$$

It follows from the combination of (3.20), (3.21) and (3.22) that

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\mathbb{R}} \mathrm{d}x\ \mu_{f(H)\delta_1}(x-\varepsilon, x+\varepsilon)^q \\
&\qquad \geq \frac{1}{2^q\varepsilon}\int_J \mathrm{d}x\ \mu_{\delta_1}(x-\varepsilon, x+\varepsilon)^q \\
&\qquad \geq \frac{1}{2^q\varepsilon}\int_J \mathrm{d}x\ A(x,\varepsilon)^q - \frac{1}{2^q} C_2\, \varepsilon^{Mq-1} \\
&\qquad \geq \frac{C_1^q \varepsilon^{q-2}}{2^q}\int_J \mathrm{d}x \int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} \frac{k^q(E)\mathrm{d}E}{\|T(E,N,0)\|^{2q}} - C_2\, \frac{\varepsilon}{2^q}\,,
\end{aligned} \tag{3.23}$$

where we fixed $M = \frac{2}{q}$. Let $E \in S$. Then, if $|x - E| \leq \varepsilon/2$ and $\varepsilon < \eta$, the definition of the set $J$ implies $x \in J$. Therefore, using Fubini Theorem and integrating on $E$ only over $S$ yields (note that $|\{x, |x-E| \leq \frac{\varepsilon}{2}\} \cap J| \geq \frac{\varepsilon}{2}$)

$$\frac{1}{\varepsilon}\int_{\mathbb{R}} \mathrm{d}x\ \mu_{f(H)\delta_1}(x-\varepsilon, x+\varepsilon)^q \geq C_q\, \varepsilon^{q-1} \int_S \frac{k^q(E)\mathrm{d}E}{\|T(E,N,0)\|^{2q}} - C_2 \frac{\varepsilon}{2}. \tag{3.24}$$

for $T$ large enough, where $\varepsilon = 1/T,\ N = [T^{1+\sigma}]$. The bound (2.26) then follows from (3.19) and (3.24).

We turn to the proof of (2.28) on $\mathbb{M}(p,f,T)$ for which the following general lower bound was shown to hold:

$$\mathbb{M}(p,f,T) \geq \left\{ \frac{C_q}{|\log\varepsilon|} I_{\mu_{f(H)\psi_0}}(q,\varepsilon) \right\}^{\frac{1}{q}}, \text{ with } \varepsilon = \frac{1}{T},\ q = \frac{1}{1+p}\,. \tag{3.25}$$

The constant $C$ depends also on $\Theta_1, \Theta_2$ in the continuous case. In the lattice case, and for compactly supported measures (which is the present case) (3.25) is the main results of [BGT1]. This result (or, more exactly, a similar one) has been extended to any measure in [Tc1, Theorem 4.2]. In a continuum setting, the lower bound (3.25) is also valid, as described in [BGT2], provided the operator $\chi_N f(H)$, $f \in \mathcal{C}_c^\infty$, is shown to be Hilbert-Schmidt, with

$$\|\chi_N f(H)\|_{HS} \leq C_{\Theta_1,\Theta_2} N^{\frac{1}{2}} \ , \tag{3.26}$$

where $\chi_N$ is the spatial projection onto $[0, N]$. That $\chi_N f(H)$ is Hilbert-Schmidt is needed in [BGT1, Theorem 3.2] and [BGT2, Lemma 1], and the explicit bound (3.26) is used in [BGT1, Eq. (3.16)] and [BGT2, Eq. (31)]. In appendix A.1 we supply this bound for the general class of potentials we consider here (the argument is done in arbitrary dimension).

Now, plugging (2.26) into (3.25) yields

$$\mathbb{M}(p, f, T)^q \geq \frac{C(q)\varepsilon^{q-1}}{|\log \varepsilon|} \int_S \frac{k^q(E)\mathrm{d}E}{\|T(E, N, 0)\|^{2q}} - C_3 \tag{3.27}$$

for $T$ large enough, where $\varepsilon = 1/T$, $N = [T^{1+\sigma}]$ or $N = T^{1+\sigma}$. The statement of Theorem 2.1 follows. □

**Proof of Theorem 2.2:** Let $S$ and $f$ be as in the theorem. We thus assume that $\gamma_S = \text{Leb-essinf}_S \gamma(E) < \infty$. By definition of $\gamma_S$ for $\nu > 0$, there is a set $S_\nu \subset S$, $|S_\nu| > 0$, such that $\gamma(E) < \gamma_S + \nu$ if $E \in S_\nu$. Now it follows from the definition of $\gamma(E)$ in (2.22) that

$$\forall\, E \in S_\nu,\ \forall\, N \geq 1, \quad \|T(E, N, 0)\| \leq h(E) N^{\gamma_S + \nu}, \tag{3.28}$$

where $h(E) = \sup_n n^{-\gamma_S - \nu} \|T(E, n, 0)\| \geq 1$ is a measurable function. Note that $h(E)$ is finite for all $E \in S_\nu$. Since $S_\nu$ is a bounded set and $f = 1$ on $S_\nu$, one can apply Theorem 2.1 with the set $S = S_\nu$ and $\sigma = \nu$. First, we get from (2.26),

$$I_{\mu_{f(H)\psi_0}}(q, \varepsilon) \geq C_q \varepsilon^{q-1} N^{-2q(\gamma_S+\nu)} \int_{S_\nu} k^q(E) h^{-2q}(E)\mathrm{d}E - C_2\varepsilon. \tag{3.29}$$

Since $h$ is finite on $S_\nu$ and $k(E)$ is positive for Lebesgue-a.e. $E$, the integral in (3.29) is a positive constant depending on $p, S, \nu$. Finally, since $N = [T^{1+\nu}]$ and $\varepsilon = T^{-1}$, we obtain from (3.29) for any $\nu > 0$, and for all $T > 0$,

$$I_{\mu_{f(H)\psi_0}}(q, \varepsilon) \geq C_{p,\nu} \left(\frac{1}{\varepsilon}\right)^{p-(1+\nu)(2\gamma_S+2\nu)} \ - \ C_2\, \varepsilon\, . \tag{3.30}$$

By definition of the generalized dimension $D^-_{\mu_{f(H)\psi_0}}(q)$, we get (2.31).

We turn to the moments $\mathbb{M}(p, f, T)$. We combine (3.30) with (3.25) to get, for any $\nu > 0$,

$$\mathbb{M}(p, f, T) \geq C(p, \nu) T^{p-2\gamma_S-\nu} \ - \ C_3 \tag{3.31}$$

with suitable new constants. In particular we get

$$\beta^-(p, f) \geq D^-_{\mu_{f(H)\psi_0}}\left(\frac{1}{p+1}\right) \geq 1 - \frac{2\gamma_S}{p}. \tag{3.32}$$

Pick now $E \in \mathbb{R}$ such that $\overline{\gamma}(E) < \infty$. Then for any bounded open interval $I \ni E$, $\gamma_I < \infty$, and it follows from the definition of the transport exponents in (2.23) that $\beta^-(p, I) \geq 1 - \frac{2\gamma_I}{p}$. Since this is true for all $I \ni E$, we get with the definition (2.25) that $\beta^-(p, E) \geq 1 - \frac{2\overline{\gamma}_E}{p}$. □

**Proof of Theorem 2.3:** We repeat the proof of Theorem 2.2 above but with the subsequence of scale $N_i$, and thus the subsequence of time $T_i = N_i^{(1+\sigma)^{-1}}$. It leads to upper limits rather than lower limits. □

**Proof of Corollary 2.1:** By hypothesis, for any $\nu > 0$ there exists a sequence $(n_i)_{i\in\mathbb{Z}^+}$ such that Leb-essinf$_S\|T(E, n_i, 0)\|^{-2} > n_i^{-(1+\alpha+\nu)}$. Thus for any $i \in \mathbb{Z}^+$, there exists a set $S_i \subset S$ of full Lebesgue measure in $S$: $|S \setminus S_i| = 0$, such that for all $E \in S_i$, one has $\|T(E, n_i, 0)\|^{-2} > n_i^{-(1+\alpha+\nu)}$. Consider $\tilde{S} = \cap_i S_i$: it has full Lebesgue measure in $S$: $|S \setminus \tilde{S}| = 0$. By construction we have that for any $E \in \tilde{S}$ and for any $i \in \mathbb{Z}^+$, $\|T(E, n_i, 0)\| < n_i^{\frac{1}{2}(1+\alpha+\nu)}$. Now Theorem 2.3 applies to the sequence $(n_i)_{i\in\mathbb{Z}^+}$ and the set $\tilde{S}$. □

# 4 An analysis of the wave-packets spreading

In this section we shall introduce and study some quantities related to the spreading of wave packets. In particular we shall make rigourous the idea that the behaviour of $\beta^\pm(p, f)$ as $p$ goes to zero is governed by the essential part of the wave packet, while the behaviour of $\beta^\pm(p, f)$ as $p$ goes to infinity is governed by its fastest part. In the sequel we restrict ourselves to the discrete Hilbert space $\ell^2(\mathbb{Z}^d)$, but the analysis can be carried over to the $L^2(\mathbb{R}^d)$ case.

Let $\psi$ be some normalized initial state, $\|\psi\| = 1$ (think about $\psi = f(H)\psi_0$, $f$ smooth and compactly supported). For any $\alpha \geq 0$, $T \geq 1$ let us consider the function

$$P_\psi(\alpha, T) = \sum_{|n|\geq T^\alpha - 1} \langle|\psi(t, n)|^2\rangle_T\,, \quad \langle|\psi(t, n)|^2\rangle_T = \frac{2}{T}\int_0^\infty \mathrm{e}^{-2t/T}\left|\langle\delta_n\,,\,\mathrm{e}^{-iHt}\psi\rangle\right|^2 \mathrm{d}t,$$

where $(\delta_n)_{n\in\mathbb{Z}^d}$ denotes the canonical basis of $\ell^2(\mathbb{Z}^d)$; in other terms

$$P_\psi(\alpha, T) = \frac{2}{T}\int_0^\infty \mathrm{e}^{-2t/T}\left\|\chi_{|n|\geq T^\alpha - 1}\mathrm{e}^{-iHt}\psi\right\|^2 \mathrm{d}t,$$

The sum is defined so that $P(0, T) = 1$ for all $T \geq 1$. Define now, for $\alpha \in [0, +\infty[$, two exponents taking values in $[0, +\infty]$:

$$S_\psi^-(\alpha) = \limsup_{T\to+\infty} \frac{\log 1/P_\psi(\alpha, T)}{\log \mathrm{T}} = -\liminf_{T\to+\infty} \frac{\log P_\psi(\alpha, T)}{\log \mathrm{T}}\,, \tag{4.1}$$

and

$$S_\psi^+(\alpha) = \liminf_{T\to+\infty} \frac{\log 1/P_\psi(\alpha, T)}{\log \mathrm{T}} = -\limsup_{T\to+\infty} \frac{\log P_\psi(\alpha, T)}{\log \mathrm{T}}\,. \tag{4.2}$$

If $P_\psi(\alpha, T) = 0$ for some $\alpha > 0$ starting from $T \geq T_0$, we define $S_\psi^\pm(\alpha) = +\infty$.

*Note that for simplicity, and most of the time, we shall drop the subscript $\psi$ and simply write $P(\alpha, T)$, $S^\pm(\alpha)$; we shall also drop the dependency in $f$ in the transport, and write $\beta^\pm(p)$.*

One immediately checks that both functions $S^-(\alpha)$ and $S^+(\alpha)$ are nondecreasing and that $S^\pm(0) = 0$. The choice of signs is such that for all $\alpha \geq 0$

$$0 \leq S^+(\alpha) \leq S^-(\alpha) \leq +\infty\,.$$

To motivate this convention note that if $S^-(\alpha) < +\infty$ for some $\alpha > 0$, then taking $\delta > 0$, one has for *all* $T$ large enough

$$P(\alpha, T) \geq T^{-S^-(\alpha)-\delta} \tag{4.3}$$

For $S^+(\alpha)$ similar bound holds on some sequence of times. The consistency of this convention will become even clearer in the sequel.

The bound (4.3) also illustrates how the functions $S^\pm(\alpha)$ control the power decaying tails of the wave packet. These tails are important when considering the moments of position operator as shown by the following proposition.

**Proposition 4.1.** *One has*

$$\beta^\pm(p) \geq \sup_{\alpha\geq 0}\left(\alpha - \frac{S^\pm(\alpha)}{p}\right) = \frac{1}{p}(S^\pm)^\sharp(p)\,, \quad p > 0,$$

*where $g^\sharp$ denotes the Legendre transform of $g$: $g^\sharp(p) = \sup_\alpha(p\alpha - g(\alpha))$.*

**Proof of Proposition 4.1:** The proof is quite immediate and follows from (4.3) and its equivalent with $S^+(\alpha)$ and time sequences. For instance, if $S^-(\alpha) < +\infty$ for some $\alpha > 0$, then it follows from (4.3) that for any $\delta > 0$ and $T$ large enough,

$$\mathbb{M}(p, f, T) = \sum_n \langle n\rangle^p \langle|\psi(t, n)|^2\rangle_T \geq (T^\alpha - 1)^p P(\alpha, T) \geq C T^{p\alpha - S^-(\alpha)-\delta}\,. \tag{4.4}$$

Thus, $\beta^-(p, f) \geq \alpha - \frac{S^-(\alpha)}{p}$ . If $S^-(\alpha) = +\infty$, the same lower bound remains trivially true. □

When considering the functions $S^\pm(\alpha)$, two couples of numbers are of particular interest:

$$\alpha_l^\pm = \sup\{\alpha \geq 0 \mid S^\pm(\alpha) = 0\}\,, \quad \alpha_u^\pm = \sup\{\alpha \geq 0 \mid S^\pm(\alpha) < +\infty\}\,. \tag{4.5}$$

Since $S^\pm(\alpha)$ are non decreasing functions,

$$0 \leq \alpha_l^\pm \leq \alpha_u^\pm \leq +\infty\,.$$

Using (4.3), one can interpret $\alpha_l^\pm$ as the (lower and upper) rates of propagation of the *essential* part of the wave packet, and $\alpha_u^\pm$ as the rates of propagation of the *fastest* (polynomially small) part of the wave packet. As pointed out in [DT], the following lower bounds holds:

$$\alpha_l^- \geq \frac{1}{d}\dim_H(\mu_\psi), \quad \alpha_l^+ \geq \frac{1}{d}\dim_P(\mu_\psi), \tag{4.6}$$

where $\dim_H$ and $\dim_P$ denote the Hausdorff and the packing dimensions of the spectral measure $\mu_\psi$ respectively. These bounds are a consequence of Corollary 4.2 below, but they follow also from classical proofs of [G], [La], [GSB1].

Of course (4.6) does not say anything in presence of pure point spectrum. One may think that pure point spectrum always implies $\alpha_l^\pm = 0$! For it is well known, from the RAGE theorem, that for any $\psi$ belonging to the point spectrum subspace,

$$\lim_{R\to+\infty} \sup_t \sum_{|n|\geq R} |\psi(t,n)|^2 = 0,$$

so that $\lim_{T\to+\infty} P(\alpha,T) = 0$ for all $\alpha > 0$. But it does not imply that $S^\pm(\alpha) = +\infty$. Indeed, for some $\alpha > 0$, $P(\alpha,T)$ may tend to 0 slowly, for instance like $(\log T)^{-a}$, $a > 0$. In which case $S^-(\alpha) = 0$ and thus $\alpha_l^- > 0$. A similar behavior for time sequences leads to $\alpha_l^+ > 0$, as illustrated by the perturbed almost Mathieu model of [DR+2], revisited in the present paper, where the spectrum is pure point, but $\alpha_l^+ = 1$ (a consequence of Theorem 5.5 and Corollary 4.2).

It follows from Proposition 4.1 that $\beta^\pm(p) \geq \alpha_l^\pm$, and with (4.6) one recovers the well known Guarneri's type lower bounds [G, C, La, GSB1]. Since $\beta^\pm(p)$ are nondecreasing, one gets

$$\beta^\pm(0^+) \equiv \lim_{p\to 0} \beta^\pm(p) \geq \alpha_l^\pm. \tag{4.7}$$

We will show that, unlike $\dim_H(\mu_\psi)$ and $\dim_P(\mu_\psi)$, the numbers $\alpha_l^\pm$ exactly characterize $\beta^\pm(0+0)$. In a similar way, one easily derives from Proposition 4.1 that

$$\beta^-(\infty) \equiv \lim_{p\to+\infty} \beta^-(p) \geq \alpha_u^-, \tag{4.8}$$

and we will show that the numbers $\alpha_u^\pm$ exactly characterize these limits $\beta^\pm(\infty)$. This makes rigourous the idea that the behaviour of $\beta^\pm(p,f)$ for small $p$ is governed by the essential part of the wave packet, while the behaviour of $\beta^\pm(p,f)$ for large $p$ is governed by its fastest part.

**Theorem 4.1.** *Assume that for some $\xi > 0$ (in most examples $\xi = 1$), and for all $p > 0$, there exists a constant $C_p > 0$ such that,*

$$\mathbb{M}(p,T) \leq C_p T^{p\xi}. \tag{4.9}$$

*Then $0 \leq \alpha_l^\pm \leq \alpha_u^\pm \leq \xi$, and*

$$\beta^\pm(p) \leq \inf_{\alpha\in(\alpha_l^\pm,\alpha_u^\pm)} \max\left(\alpha, \alpha_u^\pm - \frac{S^\pm(\alpha)}{p}\right). \tag{4.10}$$

*As a consequence,*

$$\beta^\pm(0^+) = \alpha_l^\pm, \quad \beta^\pm(\infty) = \alpha_u^\pm. \tag{4.11}$$

**Remark 4.1.** *(i) The lower bounds $\beta^\pm(0^+) \geq \alpha_l^\pm$ and $\beta^\pm(\infty) \geq \alpha_u^\pm$ do not require Condition (4.9).*
*(ii) The upper bound (4.10) is the first non linear and general upper bound for dynamics of quantum systems. In [Tc2], the behaviour of $P(\alpha,T)$ is explicitly obtained for the sparse potential model of [JL, CM], and one can compute $S^-(\alpha)$. Combining (4.10) with this analysis exactly yields the upper bound obtained in Combes-Mantica [CM] for $p \leq 2$, and for all $p$ in Tcheremchantsev [Tc2].*

**Proof of Theorem 4.1:** The lower bounds (4.7) and (4.8) are already proved. Next, since (4.9) implies $\beta^{\pm}(\infty) \le \xi$, the bound $\alpha_l^{\pm} \le \alpha_u^{\pm} \le \xi$ follows. We turn to the upper bounds, and in first place to (4.10) for $\beta^+(p,f)$. For any given $\alpha \in (\alpha_l^+, \alpha_u^+)$ and $\delta, \nu > 0$ with $\alpha_u^+ + \nu \le \xi + \delta$, we set:

$$\begin{aligned}
&\mathbb{M}(p,T) &(4.12)\\
&= \left( \sum_{|n|<T^\alpha - 1} + \sum_{T^\alpha - 2 \le |n| \le T^{\alpha_u^+ + \nu}} + \sum_{T^{\alpha_u^+ + \nu} \le |n| \le T^{\xi+\delta}} + \sum_{|n|>T^{\xi+\delta}} \right) \langle n \rangle^p \langle |\psi(t,n)|^2 \rangle_T \\
&\equiv \; R_1 + R_2 + R_3 + R_4, &(4.13)
\end{aligned}$$

Note that if $\alpha_u^+ = \xi$ we do not even need the term $R_3$. Clearly,

$$R_1 \le (T^\alpha - 1)^p \le C T^{p\alpha}, \tag{4.14}$$

and, for $T$ large enough,

$$R_2 \le T^{p(\alpha_u^+ + \nu)} P(\alpha, T) \le C T^{p(\alpha_u^+ + \nu) - (S^+(\alpha) - \nu)}. \tag{4.15}$$

Next, we estimate $R_3$ by $R_3 \le T^{p(\xi+\delta)} P(\alpha_u^+ + \nu, T)$. Since $S^+(\alpha_u^+ + \nu) = +\infty$, for any $A > 0$ for all $T$ large enough $P(\alpha_u^+ + \nu, T) \le T^{-A}$. Therefore, choosing $A = p(\xi + \delta)$ leads to, for $T$ large enough,

$$R_3 \le C\,. \tag{4.16}$$

For any $m > 0$, (4.9) implies

$$R_4 \le T^{-m(\xi+\delta)} \sum_{|n|>T^{\xi+\delta}} \langle n \rangle^{p+m} \langle |\psi(t,n)|^2 \rangle_T \le C_{p+m} T^{(p+m)\xi - m(\xi+\delta)}.$$

Taking $m > \frac{p\xi}{\delta}$, we see that

$$R_4 \le C(p,\delta). \tag{4.17}$$

It follows directly from (4.14)-(4.17) that $\beta^+(p) \le \max(\alpha, \alpha_u^+ + \nu - (S^+(\alpha) - \nu)/p)$ for all $\nu > 0$ and $\alpha \in [\alpha_l^+, \alpha_u^+]$. The result follows. The proof for $\beta^-(p)$ is similar.

Now, notice that (4.10) immediately implies $\beta^{\pm}(\infty) \le \alpha_u^{\pm}$. To prove that $\beta^+(0^+) \le \alpha_l^+$, assume that $\alpha_l^+ < \alpha_u^+$ (otherwise there is nothing to prove: $\beta^+(0^+) \le \beta^+(\infty) \le \alpha_u^+ = \alpha_l^+$), and pick $\alpha = \alpha_l^+ + \nu < \alpha_u^+$ with small $\nu > 0$. Then $S^+(\alpha) > 0$, and taking $p$ small enough one has $\alpha \ge \alpha_u^+ - S^+(\alpha)/p$, therefore (4.10) implies $\beta^+(p) \le \alpha$. It follows that $\beta^+(0^+) \le \alpha_l^+$. The proof for $\beta^-(0^+)$ is similar. □

As an immediate corollary, we can characterize quantum systems with an homogenous dynamical behaviour, namely systems where the transport exponents are constant: $\beta^{\pm}(p) = \beta^{\pm}$ for all $p > 0$.

**Corollary 4.1.** *Under the condition (4.9), $\beta^-(p) = \beta^-$ for all $p > 0$, resp. $\beta^+(p) = \beta^+$ for all $p > 0$, iff $\alpha_l^- = \alpha_u^-$, resp. iff $\alpha_l^+ = \alpha_u^+$.*

In other terms, quantum systems with an homogenous dynamical behavior are characterized by the fact that their wave-packets travel at a unique speed (but not necessarily constant for $\alpha_l^- \ne \alpha_l^+$ is possible). Thus, in such systems wave-packets do not spread out and stay gathered.

Good candidates for quantum systems with a non homogenous dynamics are operators with: random decaying potential as in section 5.1, sparse barriers as in section 5.2, polymers [JSBS], and Fibonacci potentials [DT], while we would rather expect the Almost Mathieu model treated in section 5.3 to have a homogenous dynamics (this is already proved for the upper exponents). But it turns out that to decide whether a given operator exhibits a *non* homogenous dynamical behaviour not only requires lower bounds on the dynamics but also nontrivial upper bounds, which is known to be a challenging issue for the coming years. The sole quantum system where a non homogenous behaviour has been proved to hold is the sparse barriers potential operator of [JL, CM] treated in [Tc2]: for suitable $\psi$, $\beta^-(p)$ is non constant in $p$ and one has $\dim_H(\mu_{\delta_1}) = \alpha_l^- < \alpha_u^- = 1$ (while $\beta^+(p) = 1$ for all $p$, and thus $\alpha_l^+ = \alpha_u^+ = 1$).

As a second corollary of Theorem 4.1, combining (4.11) with the main result of [BGT1] (recalled in (1.7)), we get:

**Corollary 4.2.** *Under the condition (4.9), the following bounds hold:*

$$\alpha_u^\pm \geq \frac{1}{d} \lim_{q \to 0^+} D_{\mu_\psi}^\pm(q), \qquad \alpha_l^\pm \geq \frac{1}{d} \lim_{q \to 1^-} D_{\mu_\psi}^\pm(q), \tag{4.18}$$

*where $D_{\mu_\psi}^\pm(q)$ are generalized fractal dimensions of the spectral measure of the initial state $\psi$.*

In particular we get, with possible strict inequalities,

$$\alpha_l^+ \geq \frac{1}{d} D_{\mu_\psi}^+(1-0) \geq \frac{1}{d} \dim_P(\mu_\psi), \quad \text{and} \quad \alpha_l^- \geq \frac{1}{d} D_{\mu_\psi}^-(1-0) \geq \frac{1}{d} \dim_H(\mu_\psi) \,. \tag{4.19}$$

We end this section by extracting from Theorem 2.2 and Theorem 2.3 some information on the repartition function $S^\pm(\alpha)$ and on $\alpha_u^\pm$.

**Proposition 4.2.** *Under the conditions of Theorem 2.2, resp. Theorem 2.3, for the state $\psi = f(H)\delta_1$ the following holds:*

$$\alpha_u^- = 1, \quad \textit{and} \quad S^-(\alpha) \leq 2\gamma \textit{ for all } \alpha < \alpha_u^- \ ,$$

*resp. $\alpha_u^+ = 1$ and $S^+(\alpha) \leq 2\gamma$ for all $\alpha < \alpha_u^+$.*

**Proof of Proposition 4.2:** That $\alpha_u^- = 1$ is immediate from (2.33) and the equality $\beta^-(\infty) = \alpha_u^-$. Now, for any $\alpha < \alpha_u^-$, $S^-(\alpha)$ is finite and thus $\max(\alpha, 1 - \frac{S^-(\alpha)}{p}) = 1 - \frac{S^-(\alpha)}{p}$ provided $p$ is large enough. One concludes by combining the lower bound (2.33) with the upper bound (4.10). The proof for the upper exponents is similar, using Theorem 2.3. □

**Remark 4.2.** *That Theorem 2.2 and Theorem 2.3 actually provide the exact value of $\alpha_u^\pm$ (i.e. $\alpha_u^\pm = 1$) suggests that the lower bounds supplied by these theorems indeed take into account the fastest part of the wave packet represented by $\alpha_u^\pm$ (strictly speaking, to be sure that is it true, one should prove that $\alpha_l^\pm < 1$ for the considered models).*

# 5 Applications

## 5.1 Application to random operators with decaying potential

Let us consider the Schrödinger Operator with random decaying potential on $\ell^2(\mathbb{Z}^+)$,

$$(H_\omega u)(n) = u(n+1) + u(n-1) + V_\omega(n)u(n), \tag{5.1}$$

with the convention $u(-1) = 0$ and

$$V_\omega(n) = \frac{\lambda}{\sqrt{n}}\, a_\omega(n). \tag{5.2}$$

Here the $a_\omega(n)$ are i.i.d. random variables with a bounded distribution $\mu$ (not necessarily smooth), such that the expectation $\mathrm{Exp}(a_\omega) = 0$ and $\mathrm{Exp}(a_\omega^2) = 1$. We denote by $\mathbf{P}$ the measure on the probability space, namely $\mathbf{P} = \prod_{n\geq 1} \mu$. We specified the form of the potential as in (5.2) in order to simplify the exposition. However, our result is valid under the more general condition described in [KLS, Section 8], for $\alpha = \frac{1}{2}$.

Under the condition that $a_\omega(1)$ has an absolute continuous distribution, the following was shown in [KLS], [KL]:
1. If $|\lambda| \geq 2$, then the spectrum of $H_\omega$ is $\mathbf{P}$-a.s. pure point with polynomially decaying eigenfunctions.
2. If $|\lambda| < 2$, then the spectrum of $H_\omega$ is $\mathbf{P}$-a.s. pure point on $E: \sqrt{4-\lambda^2} < |E| \leq 2$ and singular continuous on $E: \ |E| < \sqrt{4-\lambda^2}$. Moreover, in the case of singular continuous spectrum, the local Hausdorff dimension of the spectral measure $\mu_{\delta_1}$ at energy $E$ is given by $(4-E^2-\lambda^2)/(4-E^2)$.
3. Suppose that $|\lambda| < 2$ and $\psi$ is such that $P_c\psi(\omega) \neq 0$ (where $P_c$ is the projector on the continuous spectrum of $H_\omega$). Then for this initial state moments are almost ballistic. In particular, in our notations, $\beta^-(p, E) = 1$ for all $E: \ |E| < \sqrt{4-\lambda^2}$.

Note that [KLS], [LS] yield no dynamical results in the pure point regime. If the distribution of $a_\omega(1)$ is not absolutely continuous, there are no results (spectral or dynamical) at all. In the present paper we fill this gap, by applying Theorem 2.2.

We denote by $\mathbb{M}_\omega(p, f, T)$ and by $\beta_\omega^\pm(p, E), \beta_\omega^\pm(E)$ the moments and transport exponents of $H_\omega$.

**Theorem 5.1.** *Let $H_\omega$ be the operator introduced in (5.1)-(5.2). For $\mathbf{P}$ a.e. $\omega$ the following holds: for any $f \in \mathcal{C}_0^\infty(\mathbb{R})$, $f \geq 0$, $f = 1$ on some compact subinterval $J \subset (-2, 2)$, for any $\nu > 0$, there exists a finite constant $C_\omega(p, J, \nu) > 0$, such that for all sufficiently large $T$*

$$\mathbb{M}_\omega(p, f, T) \geq C_\omega(p, J, \nu) T^{p-2\gamma_J-\nu}, \tag{5.3}$$

*for $p > 2\gamma_J + \nu$ with $\gamma_J = \inf_{E\in J}\{\lambda(8-2E^2)^{-1}\}$. As a consequence for any $E \in (-2, 2)$,*

$$\beta_\omega^-(p, E) \geq 1 - \frac{\lambda}{p(4-E^2)}, \quad \textit{and thus} \quad \beta_\omega^-(\infty, E) = 1. \tag{5.4}$$

**Proof of Theorem 5.1:** Such a model fits into the framework of [KLS], so that the following holds [KLS] (Theorem 8.2):

$$\text{For a.e } \omega, \text{ for a.e } E, \quad \lim_{n\to\infty} \frac{\log \|T_\omega(E, n, 0)\|}{\log n} = \frac{\lambda}{8-2E^2} = \gamma(E). \tag{5.5}$$

Now Theorem 5.1 follows by application of Theorem 2.2. $\square$

If $\lambda < 2$ then **P**-a.s. $H_\omega$ exhibits a spectral transition from pure point ($\sqrt{4-\lambda^2} \leq |E| \leq 2$) to singular continuous spectrum ($|E| \leq \sqrt{4-\lambda^2}$) [KLS]. As Theorem 5.1 shows, this spectral transition disappears if one turns to dynamics and take $\beta^-(\infty, E)$ as a (weak) indicator of the dynamical behavior of the quantum system: $\beta^-(\infty, E)$ remains equal to 1 everywhere on $(-2, 2)$. We moreover believe that the local transport exponents of order $p$, $\beta^-(p, E)$, should increase continuously as $E$ varies from the edge of $[-2, 2]$ to its center, although our result is not sharp enough to prove this. Note that this provides a new example of Schrödinger operator with pure point spectrum and nontrivial transport. The first such example of what has been seen as a "pathological" behavior has been given by Del Rio, Jitomirskaya, Last, Simon in [DR+2]. It yielded the interesting question of what should be called *localization* [DR+1][DR+2]. The operator presented in the present article with $\lambda < 2$ raises another issue (related to the first one though), namely: what should be called a *transition*. This question is also discussed in [GK2, GK3].

We point out that the result of the Theorem does not depend on the absolute continuity of the law of $a_\omega(1)$, unlike the spectral result in [KLS] and dynamical results in [KL]. It is an explicit example illustrating an advantage of Theorem 2.2 in that it yields directly dynamical lower bounds. If $\gamma(E) \leq 1/2$, i.e. if $E$ lies in the singular continuous part of the spectrum then Theorem 5.1 provides a new result only if the distribution of $a_\omega(1)$ is not absolutely continuous.

A continuous analog of this result is as follows. Assume that $g(x) \in C_0^\infty(0, 1)$. Let $a_n(\omega)$ be i.i.d. random variables satisfying same conditions as before. Let

$$H_\omega = -\frac{d^2}{dx^2} + \lambda \sum_{n=1}^{\infty} a_n(\omega) n^{-1/2} g(x-n) \tag{5.6}$$

satisfying boundary condition $\cos\theta u(0) + \sin\theta u'(0) = 0$. Denote by $\hat{g}$ the Fourier transform of a function $g$.

**Theorem 5.2.** *Let $H_\omega$ be the operator introduced in (5.6). For* **P** *a.e. $\omega$, the following holds: for any $f \in C_0^\infty(\mathbb{R})$, $f \geq 0$, $f = 1$ on some compact subinterval $J \subset (0, \infty)$, for any $\nu > 0$, there exists a finite constant $C_\omega(p, J, \nu) > 0$, such that for all sufficiently large $T$*

$$\mathbb{M}_\omega(p, f, T) \geq C_\omega(p, J, \nu) T^{p - 2\gamma_J - \nu}, \tag{5.7}$$

*for $p > 2\gamma_J + \nu$ with $\gamma_J = \inf_{E \in J}\{\lambda (8E)^{-1} |\hat{g}(\sqrt{E})|^2\}$. As a consequence for any $E > 0$,*

$$\beta_\omega^-(p, E) \geq 1 - \frac{\lambda |\hat{g}(\sqrt{E})|^2}{4pE}, \quad \textit{and thus} \quad \beta_\omega^-(\infty, E) = 1. \tag{5.8}$$

Similarly to Theorem 5.1, this result follows from Theorem 2.2 and Theorem 9.2 of [KLS]. In [KLS], it was shown that

$$\text{For a.e } \omega, \text{ for a.e } E, \quad \lim_{n\to\infty} \frac{\log \|T_\omega(E, n, 0)\|}{\log n} = \frac{\lambda |\hat{g}(\sqrt{E})|^2}{8E} = \gamma(E). \tag{5.9}$$

Notice that while the spectral conclusions can be drawn only for a.e. $\theta$ [KLS], we obtain dynamical bounds for all boundary conditions. Moreover for a.e. $\theta$ the spectrum is pure point for $E > 0$ small enough, and singular continuous if $E$ is large enough, while there is no transport transition in the (weak) sense that $\beta_\omega^-(\infty, E) = 1$ for all $E > 0$. Note that to the best of our knowledge it is the first continuous Schrödinger operator with coexistence of point spectrum and transport.

### 5.2 Application to discrete sparse potentials

We first consider bounded barriers and provide an application of Theorem 2.2. We then propose a model with high barriers where Theorem 2.3 yields non trivial results.

We shall denote by $x_n > 0$ the location of the $n^{\text{th}}$ barrier and by $h_n \geq 0$ its height. The potential then has the form

$$V = \sum_{n=1}^{\infty} h_n \delta_{x_n}$$

acting on $\ell^2(\mathbb{Z}^+)$. We first consider the case where $|h_n| \leq a$ for all $n \geq 1$ and for some $a > 0$, and to fix the ideas we require

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \alpha \in [0, 1[\,. \tag{5.1}$$

In particular the case of barriers of same height, $h_n = a$ for all $n$, is of interest.

If $\alpha = 0$ then the potential is quite sparse; this case has been studied in detail starting from the work of Pearson [P1]. In particular, if $\alpha = 0$, Kiselev, Last and Simon [KLS] proved that the spectrum is purely singular iff $\sum_n h_n^2 = \infty$, provided $h_n \to 0$, and Krutikov, Remling [KrR] extended this result to the bounded case $\sup_n h_n < \infty$. It is moreover not very difficult to check, using either Jitomirskaya-Last's version of subordinacy theory [JL], or Proposition 2.1 that the spectral measure is 1 dimensional: $\dim_H(\mu) = 1$. Thus from the transport point of view, using Guarneri's argument as in [La], one gets quasi-ballistic transport in the energy range $(-2, 2)$. It follows that $\beta^-(p, E) = 1$ for any $E \in (-2, 2)$. Note that since one actually shows that $\gamma(E) = 0$ on $(-2, 2)$, with $\gamma(E)$ as in (2.22), $\beta^-(p, E) = 1$ can also be derived from Theorem 2.2.

The situation gets more interesting if $\alpha$ is not zero. It implies that for any small $\eta > 0$, and for $n \geq n_0$ large enough $x_{n_0}(\alpha + \eta)^{-(n-n_0)} \leq x_n \leq x_{n_0}(\alpha - \eta)^{-(n-n_0)}$. It is thus enough to treat the case

$$x_n \geq \left(\frac{1}{\alpha}\right)^n ,$$

which is the only assumption on $x_n$ we henceforth make and which is more general than (5.1). One can easily see [Z] that under conditions $\limsup x_n/x_{n+1} < 1$ and $h_n \to 0$, the spectral measure is again one-dimensional and thus the transport is quasi-ballistic. Thus, the interesting case is that of bounded $h_n$ which do not go to 0. For such model with particular choice $x_n = \gamma^n$, $\gamma \geq 2$, $h_n = v \neq 0$, Zlatos has recently shown [Z] that for some values of $\gamma, v$ the spectral measure has fractional Hausdorff dimension. We shall obtain dynamical lower bounds for this model in full generality.

For $E \in (-2, 2)$ the transfer matrix of the free Laplacian is similar to a rotation. We shall denote by $C(E)$ the constant coming from the diagonalization of the matrices, so that, for any $k \geq 0$,

$$\left\| \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}^k \right\| \leq C(E). \tag{5.2}$$

Note that $C(E)$ explodes at $E \to \pm 2$, but it is continuous in $E$ and thus remains uniformly bounded on any compact subset of $(-2,2)$. The sparseness of the potential then implies that, for any $E \in (-2,2)$,

$$\begin{aligned} \|T(E,N,0)\| &\leq C(E)^{n+1}\prod_{j=1}^{n}(h_j+3) \leq C(E)\left(C(E)(a+3)\right)^n \\ &\leq C(E)(x_n)^{\frac{\ln(C(E)(a+3))}{\ln(1/\alpha)}} \leq C(E)N^{\gamma(E)}\,, \end{aligned} \tag{5.3}$$

if $x_n \leq N < x_{n+1}$, where

$$\gamma(E) = \frac{\ln(C(E)(a+3))}{\ln(1/\alpha)}\,. \tag{5.4}$$

One can observe that in the region where $\gamma(E) < 1/2$, The Jitomirskaya-Last method yields positive local Hausdorff dimension and thus nontrivial dynamical lower bound like $\beta^-(p,E) \geq 1 - 2\gamma(E)$. We can prove more general bounds as follows.

**Theorem 5.3.** *Let $(x_n)_{n\geq 1}$ be a sequence positive integers so that $x_n \geq \alpha^{-n}$, with $\alpha \in ]0,1[$, and let $(h_n)_{n\geq 1}$ be a sequence of reals so that $0 \leq h_n \leq a$ for some $a > 0$. Consider $H = -\Delta + \sum_{n\geq 1} h_n\delta_{x_n}$ acting on $\ell^2(\mathbb{Z}^+)$ with any boundary condition. The following holds: For any $E \in (-2,2)$,*

$$\beta^-(p,E) \geq 1 - \frac{2\gamma(E)}{p}, \qquad \text{and thus} \qquad \beta^-(\infty,E) = 1\,,$$

*where $\gamma(E)$ is defined in (5.4).*

**Proof of Theorem 5.3:** The Theorem follows immediately from (5.3), Theorem 2.2 and the continuity of $\gamma(E)$ in $(-2,2)$. $\square$

Once again we stress that the above result is valid regardless the nature of the spectrum of $H$. From the spectral point of view, we believe that the situation could change dramatically as one plays with the parameters of the models: $\alpha$, $a$ and $(h_n)$. But it may be very hard to determine the precise nature of the spectrum.

We turn to the second class of sparse potentials to which we shall apply Theorem 2.3. In what follows, the height of the barriers $h_n$ grows to $\infty$ so that there is no a.c spectrum [SiSp].

**Theorem 5.4.** *Let $(h_n)_{n\geq 0}$ be a sequence of nonnegative reals with $\lim_{n\to\infty} h_n = \infty$. Pick $\alpha > 0$. For $n \geq 1$ pick $x_n$ so that $x_n \geq \prod_{i=0}^{n-1}(h_i+3)^{1/\alpha}$. Define $H = -\Delta + \sum_{n=1}^{\infty} h_n\delta_{x_n}$. Then for any $f \in \mathcal{C}_0^\infty(-2,2)$ and $\nu > 0$ there exists $C(f,p,\nu) > 0$ (and finite) so that*

$$\mathbb{M}(p,f,T_i) \geq C(f,p,\nu)T_i^{p-2\alpha-\nu}\,,$$

*for some $T_i \to \infty$, and thus for any $E \in (-2,2)$, and $p > 2\alpha$,*

$$\beta^+(p,E) \geq 1 - \frac{2\alpha}{p}, \quad \text{and therefore} \quad \beta^+(\infty,E) = 1\,.$$

*Moreover, if $\alpha < 1/2$, then $H$ has a purely singular continuous spectrum with packing dimension $\dim_P(\mu) \geq 1 - 2\alpha$ and for any $p > 0$: $\beta^+(p,E) \geq 1 - 2\alpha$.*

**Proof of Theorem 5.4:** Take $f \in \mathcal{C}_0^\infty(-2,2)$, and denote by $C_f$ the constant $C_f = \sup_{E \in \mathrm{supp} f} C(E)$, where $C(E)$ is given by (5.2) above. Pick $\nu > 0$. Since $h_n \to \infty$, we know that $(h_n+3) \geq (C_f)^{1/\nu}$ for any $n$ larger than some $n_\nu$. Following the argument described above, we have uniformly in $E \in \mathrm{supp} f$

$$\begin{aligned}
&\|T(E, x_{n+1}-1, 0)\| \\
&\qquad \leq \; (C_f)^{n+1} \prod_{j=1}^{n} (h_j+3) = (C_f)^{n_\nu+1} (C_f)^{n-n_\nu} \prod_{j=1}^{n} (h_j+3) \\
&\qquad \leq \; (C_f)^{n_\nu+1} \left( \prod_{j=1}^{n} (h_j+3) \right)^{1+\nu} \leq (C_f)^{n_\nu+1} (x_{n+1})^{\alpha(1+\nu)} ,
\end{aligned}$$

The first part of the result nows follows from Theorem 2.3.

If now $\alpha < 1/2$, then define $\varepsilon_n$ by $x_n - 1 = \varepsilon_n^{-(1+\sigma)}$, $\sigma > 0$ as in Proposition 2.1. It follows from this proposition that, for any $E \in \mathrm{supp} f$, for $n$ large enough,

$$\mu(E-\varepsilon_n, E+\varepsilon_n) \leq C(f,\nu) \varepsilon_n^{1-2\alpha(1+\nu)(1+\sigma)}.$$

Therefore, for such $E$'s,

$$\limsup_{\varepsilon \to 0} \frac{\log \mu(E-\varepsilon, E+\varepsilon)}{\log \varepsilon} \geq 1 - 2\alpha(1+\nu)(1+\sigma) \quad \forall \nu, \sigma > 0.$$

It implies that $\dim_P(\mu) \geq 1 - 2\alpha$. □

### 5.3 Transport for Almost Mathieu revisited

In this section we would like to revisit the quasi-periodic model considered by Last [La] and Del Rio, Jitomirskaya, Last Simon [DR+2] where a quasi-ballistic behavior for the upper exponent is shown ($\beta^+(p, f \equiv 1) = 1$, $p > 0$) although the measure is zero-continuous, and even pure point with exponentially localized eigenfunctions in [DR+2]. The operator is defined on $\ell^2(\mathbb{Z})$ by

$$H_{\theta,\alpha,\lambda} = -\Delta + \Lambda \cos(\pi\alpha n + \theta) + \lambda(\delta_1, \cdot)\delta_1 \,. \tag{5.1}$$

Here we take $\alpha$ irrational and $\Lambda > 2$ so that the Lyapunov exponent is positive everywhere: as a consequence the spectrum is purely singular [CFKS]. Our main purpose here is to show that the lower bound in terms of transport integrals, as provided by Barbaroux and two of us in [BGT1], actually does provide a full understanding of the mathematical phenomenon that allows such operators with singular and even pure point spectrum to exhibit quasi-ballistic transport for some time sequence. In other terms we shall show that the transport integrals behave quasi-ballistically for some sub-sequences of time and thereby that the generalized fractal dimensions $D^+(q)$ are one for $q \in (0,1)$.

We moreover complete the picture given in [DR+2] by showing that the result is valid for a dense $G_\delta$ set of irrational frequencies $\alpha$.

We shall follow closely the strategy of [La, DR+2], which is to construct periodic approximate operators. However the use of our Proposition 2.1 to get the quasi-ballistic behavior of the transport integrals makes proofs simpler. Recall the notation $I_\mu(q,\epsilon)$ (1.4).

**Theorem 5.5.** *Let $g : \mathbb{R} \to \mathbb{R}$ be any function with $\lim_{t\to\infty} g(t) = +\infty$ and $\lim_{t\to\infty} \frac{g(t)}{t} = 0$ (typically $g(t) = \log t$). There exists a dense $G_\delta$ set of irrationals $\Omega$ such that for any $\alpha \in \Omega$, for all $\theta \in [0, 2\pi[$ and $\lambda \in [0, 1]$, for any $q \in (0, 1)$, there exist a constant $C_q$ and a sequence $\varepsilon_k \to 0$ such that*

$$I_{\mu_{\delta_1}}(q, \varepsilon_k) \geq \frac{C_q}{g(\varepsilon_k^{-1})} \left(\frac{1}{\varepsilon_k}\right)^{1-q} . \tag{5.2}$$

*As a consequence $D^+_{\mu_{\delta_1}}(q) = 1$ for $q \in (0, 1)$, and thus $\beta^+(p, f \equiv 1) = 1$.*

**Remark 5.1.** *(i) As a consequence we note that for the $\lambda$'s such that the spectrum is pure point the upper generalized dimensions of $\mu_{\delta_1}$, $D^+(q)$, satisfy $D^+(q) = 1$ for $q \in (0, 1)$ and $D^+(q) \leq \dim_P(\mu_{\delta_1}) = 0$ for $q > 1$ (see [BGT3] section 4). The family of dimensions is thus discontinuous at $q = 1$.*
*(ii) Note that we are dealing with whole line operator here; therefore we apply in the proof the whole line version of Theorem 2.1. See Remark after the formulation of this theorem.*

To prove Theorem 5.5 we shall construct inductively suitable $\alpha$'s using continued fraction expansion of real numbers (like in [La, DR+2]).

**Lemma 5.1.** *Fix a rational number $\alpha_0 = p_0/q_0$. Then for every sufficiently small $\varepsilon < \epsilon(\alpha_0) > 0$ there exists $\delta(\varepsilon, \alpha_0) > 0$ such that if $|\alpha - \alpha_0| < \delta$, then for any interval $I$ containing at least one band of the periodic operator $H_{\theta,\alpha_0,\lambda=0}$ we have uniformly in $\theta \in [0, 2\pi)$ and $\lambda \in [0, 1]$,*

$$\frac{1}{\varepsilon} \int_I \mu_{\delta_1,\theta,\alpha,\lambda}(E - \varepsilon, E + \varepsilon)^q \, \mathrm{d}E \geq \frac{\varepsilon^{-1+q}}{g(\varepsilon^{-1})}, \tag{5.3}$$

*where $\mu_{\delta_1,\theta,\alpha,\lambda}$ is the spectral measure of $H_{\theta,\alpha,\lambda}$ associated to the initial state $\delta_1$.*

**Remark 5.2.** *In the particular case where $I$ is the whole spectrum, then (5.3) together with (3.19) reads $I_{\mu_{\delta_1,\theta,\alpha,\lambda}}(q, \varepsilon) \geq C_q \, g(\varepsilon^{-1})^{-1} \varepsilon^{-1+q}$.*

**Proof of Lemma 5.1:** First set $\lambda = 0$. Since $\alpha_0$ is rational the spectrum of $\sigma(H_{\theta,\alpha_0,0})$ consists of $q_0$ bands: $I_1^{\theta,\alpha_0}$, $I_2^{\theta,\alpha_0}, \cdots, I_{q_0}^{\theta,\alpha_0}$. Following [DR+2] there exists $L(\alpha_0) > 0$ independent of $\theta$ so that

$$|I_j^{\theta,\alpha_0}| \geq L(\alpha_0), \quad j = 1, 2, \cdots, q_0 \, . \tag{5.4}$$

Furthermore, uniformly in $E \in I_j^{\theta,\alpha_0}$, $j = 1, 2, \cdots, q_0$, transfer matrices are bounded uniformly in $\theta$ by some constant $C(\alpha_0)$: $\|T^\theta_{\alpha_k,\lambda=0}(E, 0, N)\|^2 \leq C(\alpha_0)$ for any $N \geq 1$. If we now let $\lambda$ vary in the compact interval $[0, 1]$ it is clear that we still have, uniformly in $\theta \in [0, 2\pi)$, $\lambda \in [0, 1]$ and $j = 1, 2, \cdots, q_0$:

$$\sup_{E \in I_j^{\theta,\alpha_0}} \sup_{N \geq 1} \|T^\theta_{\alpha_0,\lambda}(E, N, 0)\|^2 \leq C(\alpha_0). \tag{5.5}$$

We pick any $\varepsilon < \varepsilon(\alpha_0)$, where $\varepsilon(\alpha_0)$ is chosen to ensure (recall $\lim_{t\to\infty} g(t) = +\infty$)

$$\max(C(\alpha_0), L(\alpha_0)^{-1}) \leq g(\varepsilon^{-1})^{\frac{1}{1+q}} . \tag{5.6}$$

Now note that

$$|\cos(\alpha n+\theta)-\cos(\alpha_0 n+\theta)|\leq 2|\alpha-\alpha_0|n. \tag{5.7}$$

Let us pick $\sigma > 0$. As a consequence of (5.5) and (5.7), following the perturbative argument given in (3.6)-(3.7) (or see [Si1]), we get, uniformly in $\theta \in [0, 2\pi)$, $\lambda \in [0, 1]$ and $j = 1, 2, \cdots, q_0$, ,

$$\sup_{E\in I_j^{\theta,\alpha_0}}\ \sup_{1\leq N\leq \varepsilon^{-(1+\sigma)}} \|T^\theta_{\alpha,\lambda}(E,N,0)\|^2 \leq C(\alpha_0)\exp(2C(\alpha_0)|\alpha-\alpha_0|\varepsilon^{-2(1+\sigma)}) \leq 2C(\alpha_0)\,, \tag{5.8}$$

where we required that $|\alpha - \alpha_0|$ is small enough (depending on $\varepsilon$, $\sigma$, and $g$) so that $2g(\varepsilon^{-1})|\alpha - \alpha_0|\varepsilon^{-2(1+\sigma)} \leq \log 2$. Then for such $\alpha$'s, apply Proposition 2.1 with $M = 2$, which is possible with the state $\psi_0 = \delta_1$ since the spectrum of $H_{\theta,\alpha,\lambda}$ is a bounded set (so one can write $\delta_1 = f(H_{\theta,\alpha,\lambda})\delta_1$, with $f$ smooth and compactly supported). Recalling (5.6) together with $\lim_{t\to\infty} \frac{g(t)}{t} = 0$, it follows that for $\varepsilon$ small enough,

$$\mu_{\delta_1,\theta,\alpha,\lambda}(E-\varepsilon,E+\varepsilon) \;\geq\; (2C(\alpha_0))^{-1}\varepsilon-\varepsilon^2 \geq \frac{\varepsilon}{g(\varepsilon^{-1})^{\frac{1}{1+q}}}, \tag{5.9}$$

uniformly in $E \in I_j^{\theta,\alpha_0}$,$j = 1, 2, \cdots, q_0$, $\theta \in [0, 2\pi)$, $\lambda \in [0, 1]$. For such $\alpha$'s it follows from (5.9), that, uniformly in $\theta \in [0, 2\pi)$ and $\lambda \in [0, 1]$,

$$\frac{1}{\varepsilon}\int_I \mu_{\delta_1,\theta,\alpha,\lambda}(E-\varepsilon,E+\varepsilon)^q\,\mathrm{d}E \geq \frac{\varepsilon^{-1+q}}{g(\varepsilon^{-1})^{\frac{q}{1+q}}}L(\alpha_0) \geq \frac{\varepsilon^{-1+q}}{g(\varepsilon^{-1})}. \tag{5.10}$$

In the first inequality we used (5.4) and in the second (5.6). □

**Proof of Theorem 5.5:** For a fixed sequence $\gamma_k$ tending to zero, define the sets

$$A_k = \left\{\alpha \,\middle|\, \exists\varepsilon<\gamma_k : \forall\theta\in[0,2\pi),\ \lambda\in[0,1]\ \ I_{\mu_{\delta_1,\theta,\alpha,\lambda}}(q,\varepsilon) \geq \frac{1}{g(\varepsilon^{-1})}\varepsilon^{q-1}\right\}. \tag{5.11}$$

Notice the set of all $\alpha$ for which Theorem 5.5 is true contains the set $A_\infty = \cap_{k=1}^{\infty} A_k$. On the other hand, by Lemma 5.1, each of the sets $A_k$ contains a dense open set: all rational numbers $\alpha_0$ along with their small neighborhoods (which depend on $k, \alpha_0$). Therefore, $A_\infty$ contains a dense $G_\delta$ set. □

# A Appendices

## A.1 A trace estimate

Under the general hypotheses (2.17)-(2.18) on the potential, the following result follows from [KKS, Theorem 1.1] with the slight adaptation discussed in [GK2, Lemma A.4] .

**Theorem A.1 ([KKS]).** *Let $H = -\Delta + V$ on $\mathrm{L}^2(\mathbb{R}^d)$, with $V$ as in (2.17)-(2.18), and let $\nu > \frac{d}{4}$. Define $\langle X_x\rangle$ as the translation of $\langle X\rangle$ by $x \in \mathbb{Z}^d$, i.e. the multiplication operator by $\langle u - x\rangle$. There exists a constant $\mathcal{T}_{\nu,d,\Theta_1,\Theta_2}$, such that, uniformly in $x \in \mathbb{Z}^d$,*

$$\mathrm{tr}\left(\langle X_x\rangle^{-2\nu}\,\Phi_{d,\Theta_1,\Theta_2}(H)^{-1}\,\langle X_x\rangle^{-2\nu}\right) \leq \mathcal{T}_{\nu,d,\Theta_1,\Theta_2}\,, \tag{A.1}$$

*where* $\Phi_{d,\Theta_1,\Theta_2} = (E + \Theta_2 + (1-\Theta_1))^{2[[\frac{d}{4}]]}$ *and* $[[\frac{d}{4}]]$ *is the smaller integer* $> \frac{d}{4}$. *Consequently for any measurable bounded function* $f \geq 0$ *on* $\mathbb{R}$ *with compact support, one has*

$$\mathrm{tr}\left(\langle X_x\rangle^{-2\nu} f(H) \langle X_x\rangle^{-2\nu}\right) \leq \mathcal{T}_{\nu,d,\Theta_1,\Theta_2} \|f\Phi_{d,\Theta_1,\Theta_2}\|_\infty < \infty\,. \tag{A.2}$$

We prove the following

**Corollary A.1.** *Let* $H$ *be as above. For any function* $f \in \mathcal{C}_c^\infty(\mathbb{R})$, *there exists a constant* $C_{f,\Theta_1,\Theta_2,\nu,d}$ *such that, for all* $N > 0$

$$\|P_N f(H)\|_2 \leq C_{f,\Theta_1,\Theta_2,\nu,d} N^{\frac{d}{2}}\ , \tag{A.3}$$

*where* $P_N$ *is the spatial projection onto the cube* $\Lambda_N(0)$ *centered at 0 of size* $N$, *and* $\|A\|_2$ *is the Hilbert-Schmidt of* $A$.

**Proof of Corollary A.1 :** Pick $k$ s.t. $\frac{k}{2} \geq \nu + d$. It follows from Theorem A.1 that, for all $x, y \in \mathbb{Z}^d$,

$$\begin{aligned}
&\|\chi_x f(H)\chi_y\|_2^2 && \text{(A.4)}\\
&= \|\chi_x f(H)\chi_y f(H)\chi_x\|_1 \leq \|\chi_x f(H)\chi_y\|_1 \|\chi_y f(H)\chi_x\| && \text{(A.5)}\\
&\leq \|\chi_x \langle X_x\rangle^{2\nu}\| \|\|\chi_y \langle X_x\rangle^{2\nu}\| \|\| \langle X_x\rangle^{-2\nu} f(H) \langle X_x\rangle^{-2\nu} \|_1 \|\chi_y f(H)\chi_x\| && \text{(A.6)}\\
&\leq C_{f,\nu,d,\Theta_1,\Theta_2,k} \langle x-y\rangle^{-k+2\nu}\ . && \text{(A.7)}
\end{aligned}$$

Then, by taking $k$ large enough,

$$\begin{aligned}
\|\chi_x f(H)\|_2^2 &= \sum_{y\in\mathbb{Z}^d} \|\chi_x f(H)\chi_y\|_2^2 \leq C_{f,\nu,d,\Theta_1,\Theta_2,k} \sum_{y\in\mathbb{Z}^d} \langle x-y\rangle^{-k+2\nu} && \text{(A.8)}\\
&\leq C'_{f,\nu,d,\Theta_1,\Theta_2,k} && \text{(A.9)}
\end{aligned}$$

Finally, the result follows from

$$\|P_N f(H)\|_2^2 = \mathrm{tr}(f(H)P_N f(H)) \leq \sum_{x\in\mathbb{Z}^d\cap\Lambda_N(0)} \mathrm{tr}(f(H)\chi_x f(H))\ .$$

□

## A.2 An approximation Lemma

We state this approximation Lemma in a $d$-dimensional discrete setting and with abstract approximant operators. The proof in the continuous case is similar, except for the precise Combes-Thomas estimate it requires. More precisely in the $\mathrm{L}^2(\mathbb{R}^d)$ case, using the Helffer-Sjöstrand formula as below requires the explicit dependency of the constants in Combes-Thomas. We refer to [GK1] where such a version of Combes-Thomas has been derived.

**Lemma A.1.** *Let* $H_1 = H_0 + V_1$ *and* $H_2 = H_0 + V_2$, $H_0 = -\Delta$, *be operators acting on* $\ell^2(\mathbb{Z}^d)$, *such that* $V_1(x) = V_2(x)$ *for all* $|x| \leq N$ *for some* $N > 1$. *We shall assume the polynomial bound*

$$|V_1(x) - V_2(x)| \leq A\langle x\rangle^b.$$

*for all $x$ with some positive $A, b$. Let $M > 0$, $\sigma > 0$ and $f \in \mathcal{C}_c^\infty(\mathbb{R})$, $0 \le f \le 1$. There exist finite constants $C(f, M, \sigma, A, b) > 0$ and $m(M, \sigma, b)$ such that for any $\varepsilon > N^{-\frac{1}{1+\sigma}}$ and $x \in \mathbb{R}$,*

$$|\langle \delta_1, f_{x,\varepsilon}(H_1)\delta_1\rangle - \langle \delta_1, f_{x,\varepsilon}(H_2)\delta_1\rangle| \le C(f, M, \sigma, A, b)(1 + |x|)^{m(M,\sigma,b)}\varepsilon^M , \qquad \text{(A.10)}$$

*where $f_{x,\varepsilon}(y) = f((y - x)/\varepsilon)$.*
*As a consequence, if $I$ is a compact interval, there exists a finite constant $C(I, M, \sigma, A, b) > 0$ such that for any $\varepsilon > N^{-\frac{1}{1+\sigma}}$ and $x \in I$,*

$$\mu_{\delta_1}^{(H_1)}(x - \varepsilon, x + \varepsilon) \ge \mu_{\delta_1}^{(H_2)}(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}) - C(I, M, \sigma, a, b)\varepsilon^M . \qquad \text{(A.11)}$$

*where $\mu_{\delta_1}^{(H_i)}$, $i = 1, 2$, denotes the spectral measure of $H_i$ associated to the vector $\delta_1$.*

**Remark A.1.** *Relying on Gevrey functions, the recent analysis in [BGK] implies that the error term in (A.11) can actually be shown to be subexponentially small, rather than polynomially.*

We recall the Helffer-Sjöstrand formula [HS] (see also [Da, Section 2.2], [GK1, BGK]) for a self-adjoint operator $H$ and a (at least slowly) decaying smooth function $f$:

$$f(H) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\partial \tilde{f}_n}{\partial \bar{z}}(u + iv)\, R(u + iv)\, \mathrm{d}u\mathrm{d}v\,, \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\partial_u + i\partial_v\right)\,, \qquad \text{(A.12)}$$

where $z = u + iv$, $n = 1, 2, \ldots$, and $\tilde{f}_n(z)$ defined as:

$$\tilde{f}_n(u + iv) = \tau(v/\langle u\rangle)\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(u)(iv)^k, \qquad \text{(A.13)}$$

where the function $\tau$ is smooth such that $\tau(t) = 1$ for $|t| \le 1$ and $\tau(t) = 0$ for $|t| \ge 2$. A direct computation yields this useful bound:

$$\left|\frac{\partial \tilde{f}_n}{\partial \bar{z}}(u + iv)\right| \le C\left\{\sum_{k=0}^{n} \frac{1}{k!}|f^{(k)}(u)|\frac{|v|^k}{\langle u\rangle}\right\}\chi_{\mathcal{A}}(u, v) + \frac{1}{2n!}|f^{(n+1)}(u)||v|^n\chi_{\mathcal{B}}(u, v)\,, \qquad \text{(A.14)}$$

where $\mathcal{A} = \{\langle u\rangle < |v| < 2\langle u\rangle\}$, $\mathcal{B} = \{0 < |v| < 2\langle u\rangle\}$, $\chi_{\mathcal{A}}$ and $\chi_{\mathcal{B}}$ are the corresponding characteristic functions. The choice of $n$ will be made later, in (A.21).

We also recall the well-known Combes-Thomas estimate for $H = H_0 + V$: there exists a constant $\eta_d \ge 1$, depending only on the dimension $d$, such that

$$|\langle \delta_n, (H - z)^{-1}\delta_m\rangle| \le \frac{2}{\eta}\, \mathrm{e}^{-\min(\frac{\eta}{\eta_d},1)|n-m|}, \quad \eta = \mathrm{dist}(z, \sigma(H)). \qquad \text{(A.15)}$$

**Proof of Lemma A.1:**

Let $H_1$ and $H_2$ be as in the Lemma. Denote by $R_1$ and $R_2$ their respective resolvents. Hypotheses on $V_1, V_2$ imply that

$$|(V_1 - V_2)(x)| = 0 \text{ on } [-N, N]^d, \;\; |(V_1 - V_2)(x)| \le A\langle x\rangle^b \text{ outside } [-N, N]^d, \quad \text{(A.16)}$$

We first show

$$|\langle \delta_1, f_{x,\varepsilon}(H_1)\delta_1\rangle - \langle \delta_1, f_{x,\varepsilon}(H_2)\delta_1\rangle| \le C(f, M, \sigma, A, b)(1+|x|)^{m(M,\sigma,b)}\varepsilon^M, \qquad (A.17)$$

if $\varepsilon^{-1-\sigma} \le N$. Using (A.16) and the Combes-Thomas estimate (A.15), one has:

$$\begin{aligned}
\sum_{|x|\ge N+1} & |(V_1 - V_2)(x)|\,|\langle \delta_1, R_1(u+iv)\delta_x\rangle|\,|\langle \delta_x, R_2(u+iv)\delta_1\rangle| \\
&\le \sum_{x=N+1}^{+\infty} C_d A \langle x\rangle^b \langle x\rangle^{d-1} \left(\frac{2}{|v|}\right)^2 \mathrm{e}^{-2\min(\frac{|v|}{\eta_d},1)x} \\
&\le C(A,b,d)\frac{\Gamma(b+d-1)}{|v|^2 \min(\frac{|v|}{\eta_d},1)^{b+d}} \mathrm{e}^{-\min(\frac{|v|}{\eta_d},1)N}. \qquad (A.18)
\end{aligned}$$

Here $\Gamma(u) = \int_0^\infty t^u \mathrm{e}^{-t}\mathrm{d}t$. We combine (A.12), the resolvent identity, and (A.18), to get

$$\begin{aligned}
|\langle \delta_1, & f_{x,\varepsilon}(H_1)\delta_1\rangle - \langle \delta_1, f_{x,\varepsilon}(H_2)\delta_1\rangle| \\
&\le \frac{1}{\pi}\int_{\mathbb{R}^2} \left| \frac{\partial (\tilde{f_{x,\varepsilon}})_n}{\partial \bar z}(u+iv)\right| |\langle \delta_1, R_1(u+iv)(H_1-H_2)R_2(u+iv)\delta_1\rangle| \,\mathrm{d}u\mathrm{d}v \\
&\le C(A,b,d)\int_{\mathbb{R}^2} \left| \frac{\partial (\tilde{f_{x,\varepsilon}})_n}{\partial \bar z}(u+iv)\right| \frac{\mathrm{e}^{-\min(\frac{|v|}{\eta_d},1)N}}{|v|^2\min(\frac{|v|}{\eta_d},1)^{b+d}}\mathrm{d}u\mathrm{d}v. \qquad (A.19)
\end{aligned}$$

Suppose $n \ge 2+b+d$. Plug (A.14) into (A.19). The $k^{\text{th}}$ derivative of $f_{x,\varepsilon}$, $k \ge 0$, is bounded by $\|f_{x,\varepsilon}^{(k)}\|_\infty \le C_k \varepsilon^{-k}$ uniformly in $x$. Moreover note that $\mathrm{supp} f_{x,\varepsilon} \subset x + \mathrm{supp} f$ for $\varepsilon \le 1$. Divide the set $\mathcal{B}$ in two parts $\mathcal{B}_1 = \{0 < |v| \le 1\}$ and $\mathcal{B}_2 = \{1 < |v| \le \langle u\rangle\}$. Recall $\eta_d \ge 1$. As a consequence, on $\mathcal{A}$ and $\mathcal{B}_2$, one has $|v| \ge 1$ and thus $\min(\frac{|v|}{\eta_d},1) \ge \min(\frac{1}{\eta_d},1) = \frac{1}{\eta_d}$. On $\mathcal{B}_1$, $\min(\frac{|v|}{\eta_d},1) = \frac{|v|}{\eta_d}$. It yields after computations,

$$\begin{aligned}
|\langle \delta_1, & f_{x,\varepsilon}(H_1)\delta_1\rangle - \langle \delta_1, f_{x,\varepsilon}(H_2)\delta_1\rangle| \\
&\le C\sum_{k=0}^n \frac{\langle x\rangle^{k+1}\mathrm{e}^{-\frac{N}{\eta_d}}}{k!\varepsilon^k} + C\frac{\langle x\rangle^{n+1}\mathrm{e}^{-\frac{N}{\eta_d}}}{n!\varepsilon^{n+1}} + C\frac{\langle x\rangle}{n!\varepsilon^{n+1}}\int_{B_1} |v|^{n-2-b-d}\mathrm{e}^{-\frac{N}{\eta_d}|v|}\mathrm{d}v \\
&\le C\sum_{k=0}^n \frac{\langle x\rangle^{k+1}}{k!\varepsilon^k}\mathrm{e}^{-\frac{N}{\eta_d}} + C\frac{\langle x\rangle^{n+1}\mathrm{e}^{-\frac{N}{\eta_d}}}{n!\varepsilon^{n+1}} + C\frac{\Gamma(n-2-b-d)\langle x\rangle}{n!\varepsilon^{n+1}(N\eta_d^{-1})^{n-1-b-d}}. \qquad (A.20)
\end{aligned}$$

Since $N \ge \varepsilon^{-1-\sigma}$, (A.17) follows if one chooses $n$ large enough such that $(1+\sigma)(n-1-b-d) \ge M+n+1$, i.e.

$$n \ge 1+b+d+\sigma^{-1}(M+2+b+d). \qquad (A.21)$$

We turn to the second part. As in Combes-Mantica [CM], let us pick $f \in \mathcal{C}_0^\infty([-1,1])$, with $f = 1$ on $[-\frac{1}{2},\frac{1}{2}]$. It follows that for any measure $\mu$ (and thus for $\mu_{\delta_1}^{(H_1)}$ and $\mu_{\delta_1}^{(H_2)}$) one has

$$\mu(x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}) \le \int f((y-x)/\varepsilon)\mathrm{d}\mu(y) \le \mu(x-\varepsilon, x+\varepsilon).$$

Since $\int f((y-x)/\varepsilon)\mathrm{d}\mu(y) = \int f_{x,\varepsilon}(y)\mathrm{d}\mu(y) = \langle \delta_1, f_{x,\varepsilon}(H)\delta_1\rangle$, the result follows.

□

For the reader's convenience, we provide in a few lines the proof of the Combes-Thomas estimate (A.15), which holds for any type of potential in the lattice. If $A$ is an operator on $\ell^2(\mathbb{Z}^d)$, define $A_\alpha = \mathrm{e}^{\alpha l} A \mathrm{e}^{-\alpha l}$, $\alpha \in \mathbb{R}^d$. Note that $H_\alpha = \Delta_\alpha + V$, and that its resolvent satisfies $R_\alpha(z) = (H_\alpha - z)^{-1}$. Furthermore, one has, in the operator sense, $\|H_\alpha - H\| = \|\Delta_\alpha - \Delta\| \leq C_d|\alpha|$ uniformly in $|\alpha| \leq 1$, for some finite constant $C_d$ that we assume larger than 1/2 with no loss of generality. It follows from the resolvent identity that

$$\|R_\alpha(z)\| \leq \|R(z)\| + C_d|\alpha| \|R_\alpha(z)\| \|R(z)\|. \tag{A.22}$$

Let $\eta = \mathrm{dist}(z, \sigma(H))$. We impose $|\alpha| = (2C_d)^{-1}\eta$ if $\eta < \eta_d = 2C_d$ and $|\alpha| = 1$ otherwise, i.e. $|\alpha| = \min(\frac{\eta}{\eta_d}, 1)$. Note that in any case, $C_d|\alpha|\eta_d^{-1} \leq \frac{1}{2}$. As a consequence, since $\|R(z)\| \leq \eta^{-1}$, one gets $\|R_\alpha(z)\| \leq 2\|R(z)\| \leq 2\eta^{-1}$. Then, taking advantage of $R_\alpha = \mathrm{e}^{\alpha l} R(z) \mathrm{e}^{-\alpha l}$, one has, for all $\alpha$ such that $|\alpha| = \min(\frac{\eta}{\eta_d}, 1)$,

$$|\langle \delta_n, R(z)\delta_m \rangle| = \left| \mathrm{e}^{-\alpha(n-m)} \langle \delta_n, R_\alpha(z)\delta_m \rangle \right| \leq \mathrm{e}^{-|\alpha||n-m|} \|R_\alpha\|, \tag{A.23}$$

for a suitable choice of the signs of $(\alpha_1, \cdots, \alpha_d) = \alpha$. The bound (A.15) follows with $\eta_d = 2C_d \geq 1$.

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