Fluid Mechanics. Problems involving fluids are some of the hardest in the field of PDE. To understand why, you can just look at how water flows and moves in the lake, in the shower, in a creek, how plumes of smoke rise in the air, how wind swirls in a tornado. The motion is so complex and so unstable. The structures formed in the flow are persistent and yet constantly changing. How to describe all this in mathematics to be able to understand and predict how fluid moves? One of the most important equations of fluid mechanics is the Navier-Stokes equation:

$$u_t + (u \cdot \nabla)u - \kappa \Delta u = \nabla p, \;\; \nabla \cdot u = 0, \;\; u(x,0) = u_0(x). \tag{1}$$

Here u is a three-dimensional divergence free vector field, and p is pressure. The equation describes a flow of incompressible, viscous fluid. It is applied widely in physics and engineering. The equation is usually considered in two or three dimensions. If $\kappa = 0$, the equation conserves energy $\int |u|^2 \, dx$ and is then called Euler equation (which was actually derived earlier than Navier-Stokes). Both these equations are nonlinear due to the second term in (1). They are also nonlocal due to the incompressibility constraint. These properties make Euler and Navier-Stokes equations some of the hardest of the truly fundamental PDE to study. One of the key foundational questions for every PDE is existence and uniqueness of solutions. It is also of interest whether solutions corresponding to smooth initial data can develop singularities in finite time, and what these might mean. For these questions, satisfactory answers are available in two dimensions. Both Euler and Navier-Stokes equations with smooth initial data possess unique solutions which stay smooth forever. These results are not so old - they go back to 1960-70s. But in three dimensions, these questions are open. Only local existence and uniqueness results are known for both equations. For the Navier-Stokes equation, the question of global existence of smooth solutions vs. finite time blow up is one of the seven Clay Institute "millenium problems" which come with 1mln prize for a solution.

How can a singularity appear, how can it look? The simplest possible toy example is just the ODE $z'(t) = z^2(t)$, $z(0) = z_0$. The solution is $z(t) = \frac{z_0}{1-tz_0}$. Hence for some initial data, $z_0 > 0$, solution becomes infinite in finite time. The same phenomenon, but in different, usually much more sophisticated manifestations, appears in many PDE. Often, blow up tells us about some significant physical phenomenon, or warns about the border beyond which our equation no longer valid as a model of the process we are studying.

An even more interesting, but far out of reach, question is mathematically rigorous theory of turbulence. Turbulent flows exhibit some remarkable scaling properties, which are described to high degree of accuracy by Kolmogorov's phenomenological theory of turbulence. Very little rigorous analysis is available for this truly important problem. A short aside: Kolmogorov was a remarkable Russian mathematician who made fundamental and absolutely central contributions to many directions in mathematics. He pioneered rigorous probability theory. He has some very original and ground breaking work in Fourier analysis. He was among the first to write the reaction-diffusion equation. He initiated rigorous theory of information. He made key contribution to dynamics of Hamiltonian systems - you might have heard of KAM (Kolmogorov-Arnold-Moser) theorem. So, to formulate his theory of turbulence, he, in particular, spent several months on a research ship, traveling around Earth's oceans, taking measurements of winds and currents. He was truly an amazing guy! He is my favorite mathematician; somehow, whatever I do, I usually find that he initiated it. I should add that he also took education seriously and wrote math textbooks for the Russian middle and high school. I remember that at the time, I was not at all happy with the textbook. It seemed pretty hard.

Coming back to fluid mechanics, it is not like nothing is known about Navier-Stokes and Euler. A lot of knowledge is available, obtained by a wide range of techniques. Existence of weak (not very regular) solutions is known, certain kinds of blow up are ruled out, many stability questions are understood, many easier models of these equations have been studied. But it is also true that, most likely, some of the most fundamental, subtle and surprising properties of these equations are awaiting their discovery.

Much of modern research is on related equations of fluid mechanics that may be more approachable. Some examples are:

1. The surface quasi-geostrophic (SQG) equation

$$\partial_t \theta = (u \cdot \nabla)\theta - \kappa(-\Delta)^\alpha, \;\; u = R^\perp \theta, \;\; \theta(x,0) = \theta_0(x) \tag{2}$$

set on $\mathbb{R}^2$. Here $R^\perp \theta = (-R_2\theta, R_1\theta)$, with $R_{1,2}$ being Riesz transforms. On the Fourier side they are just multiplication by $k_{1,2}/|k|$, while on x side they are convolution operators

$$R_i\theta(x) = \int_{\mathbb{R}^2} \frac{x_i - y_i}{|x-y|^3}\theta(y)\,dy.$$

It is known that the equation has global regular solution if $\alpha \geq 1/2$. The case $0 < \alpha < 1/2$ or $\kappa = 0$ case is open. If you can solve this one you will get a nice job, surely tenure track at a good university right away!

2. The Hilbert transform model. This is like a toy model of SQG. It is set in one dimension.

$$\partial_t \theta = (H\theta)\theta_x - \kappa(-\Delta)^\alpha, \;\; \theta(x,0) = \theta_0(x), \tag{3}$$

where $H\theta$ is the Hilbert transform,

$$H\theta = P.V. \int \frac{\theta(y)}{x-y}\,dy.$$

It is known that this model has global regular solutions for $\alpha \geq 1/2$, and that it blows up for $\alpha < 1/4$. But the gap $\alpha \in [1/4, 1/2)$ is open, and it is a very interesting problem. Solving it should net you a nice postdoc.

3. The 2D Euler may be written in a way very similar to (3), but with more regular velocity $u = R^\perp(-\Delta)^{-1/2}\theta$. It is known that the Euler equation has global smooth solutions. What is not very well known is how fast the derivatives of the solutions can grow. The best upper bound is

$$\|\nabla\theta\|_{L^\infty} \leq C\exp(C\exp t), \tag{4}$$

double exponential in time. The best known example has barely superlinear growth in time. Making this gap smaller is a well known open problem. Coming up with better example where the gradient (or higher order Sobolev norms) grow faster will probably get you a PhD - depending on how much you improve the known examples. Improving (4) will get you at least a very nice postdoc, and quite possibly a nice tenure track right away.

You can find a very nice overview paper on 2D Euler, some recent advances and open question at http://www.ams.org/notices/201101/rtx110100010p.pdf. The paper mentions a different set of open problems, also quite important. Let me know if this no longer loads, I'll send you the file.

The area is currently very active. There are lots of other problems around these ones, some of which are more approachable but still good for a decent PhD.